Published by IOP Publishing for SISSA
Received: *October 2, 2008*
Accepted: *December 10, 2008*
Published: *December 22, 2008*

Iterated amplitudes in the high-energy limit

Vittorio Del Duca
Istituto Nazionale di Fisica Nucleare, Laboratori Nazionali di Frascati, 00044 Frascati (Roma), Italy
E-mail: delduca@lnf.infn.it

Claude Duhr
Inst. de Physique Théorique & Centre for Particle Physics and Phenomenology (CP3), Université Catholique de Louvain, Chemin du Cyclotron 2, B-1348 Louvain-la-Neuve, Belgium
E-mail: claude.duhr@uclouvain.be

E.W.N. Glover
Institute for Particle Physics Phenomenology, University of Durham, Durham, DH1 3LE, U.K.
E-mail: E.W.N.Glover@durham.ac.uk

Abstract: We consider the high-energy limits of the colour ordered four-, five- and six-gluon MHV amplitudes of the maximally supersymmetric QCD in the multi-Regge kinematics where all the gluons are strongly ordered in rapidity. We show that various building blocks occurring in the Regge factorisation (the Regge trajectory, the coefficient functions and the Lipatov vertex) satisfy an iterative structure very similar to the Bern-Dixon-Smirnov (BDS) ansatz. This iterative structure, combined with the universality of the building blocks, enables us to show that any two- and three-loop amplitude in multi-Regge kinematics is guaranteed to satisfy the BDS ansatz. We also consider slightly more general kinematics where the strong rapidity ordering applies to all the gluons except the two with either the largest or smallest rapidities, and we derive the iterative formula for the associated coefficient function. We show that in this kinematic limit the BDS ansatz is also satisfied. Finally, we argue that only for more general kinematics — e.g. with three gluons having similar rapidities, or where the two central gluons have similar rapidities — can a disagreement with the BDS ansatz arise.

Keywords: Supersymmetric gauge theory, Gauge Symmetry, QCD.

Contents

---

1. Introduction

Recently, Bern, Dixon and Smirnov have proposed an ansatz [1] for the colour-stripped l-loop n-gluon scattering amplitude in the maximally supersymmetric $\mathcal{N}=4$ Yang-Mills theory (MSYM), with the maximally-helicity violating (MHV) configuration for arbitrary l and n. They checked that the ansatz agrees analytically with the evaluation of the three-loop four-gluon amplitude. The ansatz has been proven to be correct also for the two-loop five-gluon amplitude, which has been computed numerically [2, 3]. The ansatz implies a tower of iteration formulæ, which allow one to determine the n-gluon amplitude at a given number of loops in terms of amplitudes with fewer loops. For example, the iteration formula for the colour-stripped two-loop MHV amplitude $m_n^{(2)}(\epsilon)$ is

$$m_n^{(2)}(\epsilon) = \frac{1}{2}\left[m_n^{(1)}(\epsilon)\right]^2 + f^{(2)}(\epsilon)\, m_n^{(1)}(2\epsilon) + Const^{(2)} + \mathcal{O}(\epsilon)\,, \tag{1.1}$$

thus the two-loop amplitude is determined in terms of a constant, $Const^{(2)}$, a known function, $f^{(2)}(\epsilon)$, of the dimensional-regularisation parameter ϵ (which is related to the cusp [4, 5] and collinear [6, 7] anomalous dimensions) and the one-loop MHV amplitude $m_n^{(1)}(\epsilon)$ evaluated to $\mathcal{O}(\epsilon^2)$.

The BDS ansatz was first predicted to fail by Alday and Maldacena [8, 9], for amplitudes with a large number of gluons in the strong-coupling limit. They claimed that the finite pieces of the two-loop amplitudes with six or more gluons would be incorrectly determined. One can characterise this statement by the quantity $R_n^{(2)}$

$$R_n^{(2)} = m_n^{(2)}(\epsilon) - \frac{1}{2}\left[m_n^{(1)}(\epsilon)\right]^2 - f^{(2)}(\epsilon)\, m_n^{(1)}(2\epsilon) - Const^{(2)}\,, \tag{1.2}$$

where $R_n^{(2)}$ may be a function of the kinematical parameters of the n-gluon amplitude, but a constant with respect to ϵ. Then the claim was that $R_n^{(2)} \neq 0$ for $n \geq 6$. This prediction was backed up by Drummond et al. [10], who considered the finite contribution to the hexagonal light-like Wilson loop at two loops. The conclusion was that either the BDS ansatz was wrong, or the equivalence between Wilson loops and scattering amplitudes did not work at two loops. The question was settled in ref. [11, 12] by the numerical calculation of $m_6^{(1)}(\epsilon)$ to $\mathcal{O}(\epsilon^2)$ and of $m_6^{(2)}(\epsilon)$, which allowed for the numerical evaluation of $R_6^{(2)}$ and showed that it was different from zero. This result also confirmed the equivalence between the scattering amplitude and the finite part of the light-like hexagon Wilson loop [13].

The question remains of how one can determine the function $R_n^{(2)}$? A direct analytical evaluation in general kinematics is currently beyond our capability: it would require the computation of the one-loop hexagon to $\mathcal{O}(\epsilon^2)$, as well as the two-loop hexagon through to $\mathcal{O}(\epsilon^0)$. Another approach is to try to constrain $R_n^{(2)}$ using some simplified kinematics, where one knows that the amplitude has certain factorisation properties. Examples include the limit where one or more of the gluons are soft or where two or more of the gluons are collinear. In this paper, we consider another limit where the kinematics is simplified - the high energy limit (HEL). For a multiparticle process there are several high energy limits that one can take, corresponding to two or more of the gluon rapidities being strongly ordered, together with constraints on the transverse momenta of the gluons. By relaxing the restriction on the gluon rapidities and transverse momenta, one can systematically return to the general kinematics. The simplest kinematics corresponds to the multi-Regge kinematics [14], where all of the produced gluons are strongly ordered in rapidity and have comparable transverse momenta. We shall start then with the simplest possible kinematics and we will show that $R_n^{(2)}$ does not contribute for any n. Then we shall consider various quasi-multi-Regge kinematics, which gradually approach the more general kinematics, with a view to determining where the function $R_n^{(2)}$ might not vanish and could therefore be constrained by the HEL.

Our paper is organised as follows. In section 2, we review the multi-Regge kinematics and discuss the Regge factorisation that tree-level (colour stripped) amplitudes obey. In section 3, we extend the Regge factorisation beyond leading order and provide a conjecture for the factorised form for the colour stripped n-gluon amplitude to all orders, both in the unphysical region, where all invariants are space-like, and in the physical region. The high-energy limits of the four-, five- and six-gluon MHV amplitudes are developed in section 4, including explicit expressions for the Regge trajectory (up to three loops), the coefficient functions (up to three loops) and the Lipatov vertex in MSYM. In section 5 we consider the BDS ansatz in the multi-Regge kinematics. By considering the four- and five-point amplitudes, we show that both the coefficient function and the Lipatov vertex satisfy an iterative structure very similar to the BDS ansatz itself.[1] This iterative structure ensures that the six-point amplitude is completely determined by known functions, and, in the multi-Regge kinematics is guaranteed to satisfy the BDS ansatz. In other words, the remainder function $R_6^{(2)}$ vanishes in the multi-Regge kinematics. Note that we disagree with Bartels *et al.* who claim that the two-loop six-point amplitude violates the Regge factorisation [15]. We trace this back to the non-commuting of the multi-Regge limit and the ϵ-expansion of the two-mass easy box as described in detail in appendix C. We derive exponentiated forms for the coefficient functions and Lipatov vertex in section 6 and prove that we recover the BDS ansatz in the multi-Regge kinematics for any number of loops. We consider other quasi-multi-Regge kinematics in section 7. In particular, we consider the slightly more general kinematics where all but two of the gluons (the two gluons with either the largest or smallest rapidities) are strongly ordered in rapidity. This quasi-multi-Regge kinematics first occurs in the five gluon amplitude and introduces a new coefficient

[1] It is well known that the l-loop Regge trajectory is directly related to $f^{(l)}(\epsilon)$.

function with two final state gluons which also satisfies an iterative structure similar to the BDS ansatz. Once again, $R_6^{(2)}$ does not contribute in this limit and we note that the conformal kinematic ratios also take a particularly simple form in this quasi-multi-Regge kinematics. Finally, in section 8 we consider more general kinematics - with three gluons having similar rapidities, or where the two central gluons have similar rapidities. These configurations first appear with four gluons in the final state. The new vertices and coefficient functions associated with these kinematics cannot be determined using the five-gluon amplitude, and require explicit knowledge of the six-gluon amplitude. We therefore cannot say anything about the sensitivity of the HEL to $R_6^{(2)}$, but note that in each of these cases, the three conformal kinematic ratios relevant for six-gluon scattering do not simplify, and take general values. We enclose appendices detailing the multi-Regge kinematics and the ϵ-expansion of the two-mass easy box one-loop integral.

2. Multi-Regge kinematics

Because in this work we make repeated use of the multi-Regge kinematics, we shall give here a short pedagogical introduction to it. We consider an n-gluon amplitude, $g_1\, g_2 \to g_3\, g_4 \cdots g_n$, with all the momenta taken as outgoing, and label the gluons cyclically clockwise. In the multi-Regge kinematics [14], the produced gluons are strongly ordered in rapidity and have comparable transverse momenta,

$$y_3 \gg y_4 \gg \cdots \gg y_n; \qquad |p_{3\perp}| \simeq |p_{4\perp}| \ldots \simeq |p_{n\perp}| \,. \tag{2.1}$$

Accordingly, we can write the Mandelstam invariants in the approximate form[2]

$$\begin{aligned} s_{12} &\simeq |p_{3\perp}||p_{n\perp}|e^{y_3-y_n}\,, \\ s_{2i} &\simeq -|p_{3\perp}||p_{i\perp}|e^{y_3-y_i}\,, \\ s_{1i} &\simeq -|p_{i\perp}||p_{n\perp}|e^{y_i-y_n}\,, \\ s_{ij} &\simeq |p_{i\perp}||p_{j\perp}|e^{|y_i-y_j|}\,. \end{aligned} \tag{2.2}$$

for $i, j = 3, \ldots, n$. We label the momenta transferred in the t-channel as

$$\begin{aligned} q_1 &= p_1 + p_n \\ q_2 &= q_1 + p_{n-1} = q_3 - p_{n-2} \\ &\vdots \\ q_{n-4} &= q_{n-5} + p_5 = q_{n-3} - p_4 \\ q_{n-3} &= -p_2 - p_3\,, \end{aligned} \tag{2.3}$$

with virtualities $t_i = q_i^2$. Then it is easy to see that in the multi-Regge kinematics the transverse components of the momenta q_i dominate over the longitudinal components,

[2] In appendices A and B, we write the invariants (2.2) and the spinor products (2.7), in terms of light-cone coordinates. Although the light-cone formulation is more convenient for performing calculations, we prefer to give here those quantities in terms of rapidities because it is physically more intuitive. The translation between light-cone coordinates and rapidities is straightforward (please see appendix A).

JHEP12(2008)097

$q_i^2 \simeq -|q_{i\perp}|^2$. In addition, $t_1 = s_{1n}$ and $t_{n-3} = s_{23}$, and we label $s = s_{12}$, and $s_1 = s_{n-1,n}$, $s_2 = s_{n-2,n-1}$, ..., $s_{n-3} = s_{34}$ for $n > 4$. Thus, the multi-Regge kinematics (2.1) become

$$s \gg s_1,\ s_2, \ldots, s_{n-3} \gg -t_1,\ -t_2, \ldots, -t_{n-3}\,, \tag{2.4}$$

with the special case $s \gg -t$ for $n = 4$. Labelling the transverse momenta of the gluons emitted along the ladder as $\kappa_1 = |p_{n-1\perp}|^2$, $\kappa_2 = |p_{n-2\perp}|^2$, ..., $\kappa_{n-4} = |p_{4\perp}|^2$, and using eq. (2.2), we can write

$$\kappa_1 = \frac{s_1\, s_2}{s_{n-2,n-1,n}} \qquad \kappa_2 = \frac{s_2\, s_3}{s_{n-3,n-2,n-1}} \qquad \cdots \qquad \kappa_{n-4} = \frac{s_{n-4}\, s_{n-3}}{s_{345}}, \tag{2.5}$$

for $n > 4$, which are known as the mass-shell conditions (B.4) for the gluons along the ladder. Eq. (2.2) also implies a relation amongst the mass-shell conditions,

$$s\, \kappa_1 \cdots \kappa_{n-4} = s_1\, s_2 \cdots s_{n-3}\,. \tag{2.6}$$

In the multi-Regge kinematics the spinor products are given by eq. (B.5)

$$\begin{aligned}
\langle 21\rangle &\simeq -\sqrt{|p_{3\perp}||p_{n\perp}|}\exp\left(\frac{y_3-y_n}{2}\right)\,,\\
\langle 2i\rangle &\simeq -i\sqrt{\frac{|p_{3\perp}|}{|p_{i\perp}|}}\,p_{i\perp}\exp\left(\frac{y_3-y_i}{2}\right)\,,\\
\langle i1\rangle &\simeq i\sqrt{|p_{i\perp}||p_{n\perp}|}\,\exp\left(\frac{y_i-y_n}{2}\right)\,,\\
\langle ij\rangle &\simeq -\sqrt{\frac{|p_{i\perp}|}{|p_{j\perp}|}}\,p_{j\perp}\exp\left(\frac{y_i-y_j}{2}\right) \qquad \text{for } y_i > y_j.
\end{aligned} \tag{2.7}$$

2.1 MHV amplitudes in multi-Regge kinematics

The colour decomposition of the tree-level n-gluon amplitude is [16]

$$\mathcal{M}_n^{(0)} = 2^{n/2}\, g^{n-2} \sum_{S_n/Z_n} \mathrm{tr}(T^{d_1}\cdots T^{d_n})\, m_n^{(0)}(1,\ldots,n)\,, \tag{2.8}$$

where d_i is the colour of a gluon of momentum p_i and helicity ν_i. The T's are the colour matrices[3] in the fundamental representation of SU(N) and the sum is over the noncyclic permutations S_n/Z_n of the set $[1,\ldots,n]$. We consider the MHV configurations $(-,-,+,\ldots,+)$ for which the tree-level gauge-invariant colour-stripped amplitudes assume the form

$$m_n^{(0)}(1,2,\ldots,n) = \frac{\langle p_i p_j\rangle^4}{\langle p_1 p_2\rangle \cdots \langle p_{n-1} p_n\rangle\langle p_n p_1\rangle}\,, \tag{2.9}$$

where i and j are the gluons of negative helicity. The colour structure of eq. (2.8) in multi-Regge kinematics is known [17–19] and will not be considered further. Here we shall

[3]We use the normalization $\mathrm{tr}(T^c T^d) = \delta^{cd}/2$, although it is immaterial in what follows.

JHEP12(2008)097

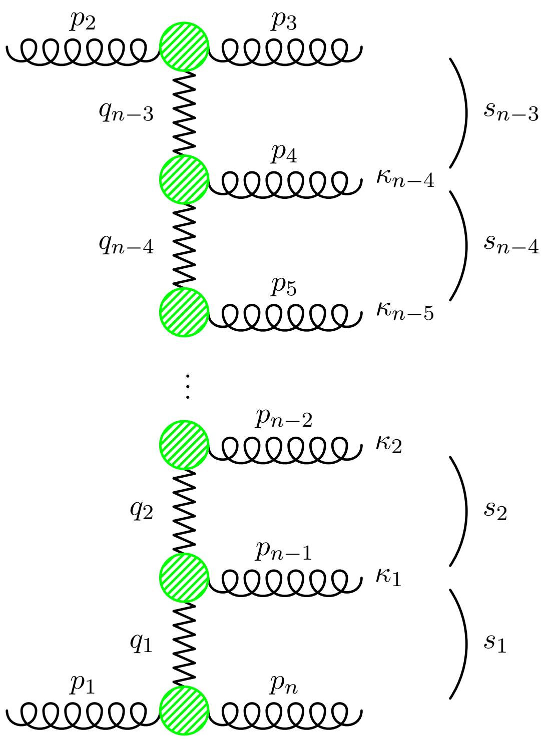

Figure 1: Amplitude in the multi-Regge kinematics. The green blobs indicate the coefficient functions (impact factors) and the Lipatov vertices describing the emission of gluons along the ladder.

concentrate on the behaviour of the colour-stripped amplitudes (2.9), which in multi-Regge kinematics has the factorised form [18]

$$m_n^{(0)}(1,2,\ldots,n) = s\left[g\,C^{(0)}(p_2,p_3)\right]\frac{1}{t_{n-3}}\left[g\,V^{(0)}(q_{n-3},q_{n-4};\kappa_{n-4})\right] \qquad (2.10)$$
$$\cdots\times\frac{1}{t_2}\left[g\,V^{(0)}(q_2,q_1,\kappa_1)\right]\frac{1}{t_1}\left[g\,C^{(0)}(p_1,p_n)\right].$$

This factorization is shown schematically in figure 1. The gluon coefficient functions $C^{(0)}$, which yield the LO gluon impact factors, are given in ref. [14] in terms of their spin structure and in ref. [18, 20] at fixed helicities of the external gluons,

$$C^{(0)}(p_2^-,p_3^+) = 1 \qquad C^{(0)}(p_1^-,p_n^+) = \frac{p_{n\perp}^*}{p_{n\perp}}, \qquad (2.11)$$

with $p_\perp = p_x + ip_y$ the complex transverse momentum. The vertex for the emission of a gluon along the ladder is the Lipatov vertex [18, 21, 22]

$$V^{(0)}(q_{j+1},q_j,\kappa_j) = \sqrt{2}\,\frac{q_{j+1\perp}^*q_{j\perp}}{p_{n-j\perp}}, \qquad (2.12)$$

with $p_{n-j} = q_{j+1} - q_j$.

JHEP12(2008)097

3. The high-energy limit of the n-gluon amplitude

The virtual radiative corrections to eq. (2.10) in the leading logarithmic (LL) approximation are obtained, to all orders in α_S, by replacing the propagator of the t-channel gluon by its reggeised form [14]. That is, by making the replacement

$$\frac{1}{t_i} \to \frac{1}{t_i}\left(\frac{s_i}{\tau}\right)^{\alpha(t_i)}, \tag{3.1}$$

in eq. (2.10), where $\alpha(t_i)$ can be written in dimensional regularization in $d = 4 - 2\epsilon$ dimensions as

$$\alpha(t_i) = g^2\, c_\Gamma \left(\frac{\mu^2}{-t_i}\right)^{\epsilon} N\,\frac{2}{\epsilon}, \tag{3.2}$$

with N colours, and

$$c_\Gamma = \frac{1}{(4\pi)^{2-\epsilon}}\,\frac{\Gamma(1+\epsilon)\,\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\,. \tag{3.3}$$

$\alpha(t_i)$ is the Regge trajectory and accounts for the higher order corrections to gluon exchange in the t_i channel. In eq. (3.1), the reggeisation scale τ is introduced to separate contributions to the reggeized propagator, the coefficient function and the Lipatov vertex. It is much smaller than any of the s-type invariants, and it is of the order of the t-type invariants. In order to go beyond the LL approximation and to compute the higher-order corrections to the Lipatov vertex (2.12), we need a high-energy prescription [23] that disentangles the virtual corrections to the Lipatov vertex from those to the coefficient functions (2.11) and from those that reggeize the gluon (3.1). The high-energy prescription of ref. [23] is given at the colour-dressed amplitude level in QCD, where it holds to the next-to-leading-logarithmic (NLL) accuracy. However, it has been shown to break down in the imaginary part of the QCD one-loop four-parton amplitude [24], in the imaginary part of the QCD one-loop five-gluon amplitude [25], and in the two-loop four-point amplitude in MSYM [26]. This is because the mismatches between the colour orderings and the multi-Regge kinematics become apparent at NLL. When the colour ordering is correctly aligned with the multi-Regge limit, the factorisation applies to NLL and beyond. In ref. [26], we showed that the high-energy prescription, applied to the colour-stripped four-point amplitude is valid up to three loops. Thus, we conjecture that in the multi-Regge kinematics a generic colour-stripped n-gluon amplitude has the factorised form,

$$\begin{aligned} m_n(1,2,\dots,n) = s\,[g\,C(p_2,p_3)]\,\frac{1}{t_{n-3}}\left(\frac{-s_{n-3}}{\tau}\right)^{\alpha(t_{n-3})}[g\,V(q_{n-3},q_{n-4},\kappa_{n-4})] \\ \cdots\times\frac{1}{t_2}\left(\frac{-s_2}{\tau}\right)^{\alpha(t_2)}[g\,V(q_2,q_1,\kappa_1)]\,\frac{1}{t_1}\left(\frac{-s_1}{\tau}\right)^{\alpha(t_1)}[g\,C(p_1,p_n)]\,, \end{aligned} \tag{3.4}$$

where we suppressed the dependence of the coefficient function and of the Lipatov vertex on the reggeisation scale τ, and on the dimensional regularisation parameters μ^2 and ϵ.

In order for the colour-stripped amplitude m_n to be real, we take eq. (3.4) in the unphysical region where the invariants are all negative,

$$s, s_1, s_2, \dots, s_{n-3}, t_1, t_2, \dots t_{n-3} < 0\,, \tag{3.5}$$

where the multi-Regge kinematics (2.4) are

$$-s \gg -s_1, -s_2, \ldots, -s_{n-3} \gg -t_1, -t_2 \ldots, -t_{n-3}\,, \tag{3.6}$$

and the on-shell condition (2.5) is

$$-\kappa_1 = \frac{(-s_1)\,(-s_2)}{-s_{n-2,n-1,n}}\,, \quad -\kappa_2 = \frac{(-s_2)\,(-s_3)}{-s_{n-3,n-2,n-1}}\,, \quad \cdots \quad -\kappa_{n-4} = \frac{(-s_{n-4})\,(-s_{n-3})}{-s_{345}}. \tag{3.7}$$

In eq. (3.4), the Regge trajectory has the perturbative expansion,

$$\alpha(t_i) = \bar{g}^2\bar{\alpha}^{(1)}(t_i) + \bar{g}^4\bar{\alpha}^{(2)}(t_i) + \bar{g}^6\bar{\alpha}^{(3)}(t_i) + \mathcal{O}(\bar{g}^8)\,, \tag{3.8}$$

with $i = 1, \ldots, n-3$, and with the rescaled coupling

$$\bar{g}^2 = g^2 c_\Gamma N\,. \tag{3.9}$$

In eq. (3.4), the coefficient functions C and the Lipatov vertex V are also expanded in the rescaled coupling,

$$C(p_i, p_j, \tau) = C^{(0)}(p_i, p_j)\left(1 + \sum_{r=1}^{s-1} \bar{g}^{2r}\bar{C}^{(r)}(t_k, \tau) + \mathcal{O}(\bar{g}^{2s})\right)\,, \tag{3.10}$$

$$V(q_{j+1}, q_j, \kappa_j, \tau) = V^{(0)}(q_{j+1}, q_j)\left(1 + \sum_{r=1}^{s-1} \bar{g}^{2r}\bar{V}^{(r)}(t_{j+1}, t_j, \kappa_j, \tau) + \mathcal{O}(\bar{g}^{2s})\right)\,.$$

with $(p_i + p_j)^2 = t_k$ where C and V are real, up to overall complex phases in $C^{(0)}$, eq. (2.11), and $V^{(0)}$, eq. (2.12), induced by the complex-valued helicity bases. Note that because several transverse scales occur, we prefer to keep the dependence on μ^2 of the trajectory, coefficient function and Lipatov vertex within the loop coefficient rather than in the rescaled coupling,

$$\bar{\alpha}^{(n)}(t_i) = \left(\frac{\mu^2}{-t_i}\right)^{n\epsilon}\alpha^{(n)}\,, \qquad \bar{C}^{(n)}(t_k, \tau) = \left(\frac{\mu^2}{-t_k}\right)^{n\epsilon} C^{(n)}(t_k, \tau)\,,$$

$$\bar{V}^{(n)}(t_{j+1}, t_j, \kappa_j, \tau) = \left(\frac{\mu^2}{-\kappa_j}\right)^{n\epsilon} V^{(n)}(t_{j+1}, t_j, \kappa_j, \tau)\,. \tag{3.11}$$

The expansion of eq. (3.4) can be written as

$$m_n = m_n^{(0)}\left(1 + \bar{g}^2\, m_n^{(1)} + \bar{g}^4 m_n^{(2)} + \bar{g}^6 m_n^{(3)} + \mathcal{O}(\bar{g}^8)\right)\,. \tag{3.12}$$

3.1 Analytic continuation of the n-gluon amplitude to the physical region

We analytically continue the high-energy prescription for the colour-stripped amplitude (3.4) to the physical region, where

$$s, s_1, s_2, \ldots s_{n-3} > 0\,, \qquad t_1, t_2, \ldots t_{n-3} < 0\,, \tag{3.13}$$

through the usual prescription $\ln(-s_j) = \ln(s_j) - i\pi$, for $s_j > 0$. Then the multi-Regge kinematics are given by eq. (2.4) and the mass-shell condition by eq. (2.5). We still use the

JHEP12(2008)097

expansions of eqs. (3.8)–(3.11), but because of the analytic continuation on $\kappa_1, \ldots, \kappa_{n-3}$ (which follows directly from the eq. (3.7) once the analytic continuation on the s-type invariants is established), in going from eq. (3.7) to eq. (2.5), the Lipatov vertices become complex,

$$\bar{V}^{(n)}(t_{j+1}, t_j, \kappa_j, \tau) = \left(\frac{\mu^2}{\kappa_j}\right)^{n\epsilon} V^{(n)}_{\rm phys}(t_{j+1}, t_j, \kappa_j, \tau)\,, \tag{3.14}$$

with

$$V^{(n)}_{\rm phys}(t_{j+1}, t_j, \kappa_j, \tau) = e^{i\pi n\epsilon}\, V^{(n)}(t_{j+1}, t_j, \kappa_j, \tau)\,. \tag{3.15}$$

4. The high-energy limit of the four-, five- and six-point MHV amplitudes

4.1 The four-point amplitude in multi-Regge kinematics

For the 4-point amplitude, $g_1\, g_2 \to g_3\, g_4$, the high-energy prescription (3.4) becomes

$$m_4(1,2,3,4) = s\,[g\, C(p_2,p_3,\tau)]\, \frac{1}{t} \left(\frac{-s}{\tau}\right)^{\alpha(t)} [g\, C(p_1,p_4,\tau)]\ . \tag{4.1}$$

In order for the colour-stripped amplitude m_4 to be real, we take it in the unphysical region where s is negative. Then the Regge kinematics are,

$$-s \gg -t\,. \tag{4.2}$$

Using the loop expansions of the Regge trajectory (3.8) and of the coefficient function (3.10), eq. (4.1) can be written as eq. (3.12) for $n = 4$. Then the knowledge of the l-loop coefficient $m_4^{(l)}$ allows one to derive the l-loop trajectory $\alpha^{(l)}$ and coefficient function $C^{(l)}(t,\tau)$. For example, the one-loop coefficient is given by

$$m_4^{(1)} = \bar{\alpha}^{(1)}(t) L +\ 2\bar{C}^{(1)}(t,\tau)\,, \tag{4.3}$$

with $L = \ln(-s/\tau)$, and $\bar{\alpha}$ and $\bar{C}$ rescaled as in eq. (3.11). The one-loop trajectory is given by eq. (3.2),

$$\alpha^{(1)} = \frac{2}{\epsilon}\,, \tag{4.4}$$

and it is the same in QCD and in MSYM. The one-loop coefficient function, $C^{(1)}$, has been computed in ref. [23, 24, 27 − 29]. In MSYM it is, to all orders in ϵ

$$\begin{aligned} C^{(1)}(t,\tau) &= \frac{\psi(1+\epsilon) - 2\psi(-\epsilon) + \psi(1)}{\epsilon} - \frac{1}{\epsilon}\ln\frac{-t}{\tau} \\ &= \frac{1}{\epsilon^2}\left(-2 - \epsilon\,\ln\frac{-t}{\tau} + 3\sum_{n=1}^{\infty}\zeta_{2n}\,\epsilon^{2n} + \sum_{n=1}^{\infty}\zeta_{2n+1}\,\epsilon^{2n+1}\right)\ . \end{aligned} \tag{4.5}$$

In fact, in the formulæ that follow we shall need $C^{(1)}(t,\tau)$ through $\mathcal{O}(\epsilon^4)$.

The two-loop coefficient of eq. (3.12) with $n = 4$ is

$$\begin{aligned} m_4^{(2)} &= \frac{1}{2}\left(\bar{\alpha}^{(1)}(t)\right)^2 L^2 + \left(\bar{\alpha}^{(2)}(t) + 2\,\bar{C}^{(1)}(t,\tau)\bar{\alpha}^{(1)}(t)\right)\, L \\ &\qquad\qquad\qquad + 2\,\bar{C}^{(2)}(t,\tau) + \left(\bar{C}^{(1)}(t,\tau)\right)^2 \\ &= \frac{1}{2}\left(m_4^{(1)}\right)^2 + \bar{\alpha}^{(2)}(t)L + 2\,\bar{C}^{(2)}(t,\tau) - \left(\bar{C}^{(1)}(t,\tau)\right)^2\ . \end{aligned} \tag{4.6}$$

JHEP12(2008)097

where in the second equality we factor out the square of the one-loop amplitude, in order to to facilitate the later comparison with the BDS ansatz. In eq. (4.6), $m_4^{(1)}$ must be known to $\mathcal{O}(\epsilon^2)$. The two-loop trajectory, $\alpha^{(2)}$, is known in full QCD [30−34]. In MSYM, it has been computed through $\mathcal{O}(\epsilon^0)$ directly [35] and using the maximal trascendentality principle [36], and through $\mathcal{O}(\epsilon^2)$ directly [26],

$$\alpha^{(2)} = -\frac{2\zeta_2}{\epsilon} - 2\zeta_3 - 8\zeta_4\epsilon + (36\zeta_2\zeta_3 + 82\zeta_5)\epsilon^2 + \mathcal{O}(\epsilon^3)\,. \tag{4.7}$$

The MSYM two-loop coefficient function has been computed through $\mathcal{O}(\epsilon^2)$ [26],

$$\begin{aligned} C^{(2)}(t,\tau) &= \frac{2}{\epsilon^4} + \frac{2}{\epsilon^3}\ln\frac{-t}{\tau} - \left(5\zeta_2 - \frac{1}{2}\ln^2\frac{-t}{\tau}\right)\frac{1}{\epsilon^2} - \left(\zeta_3 + 2\zeta_2\ln\frac{-t}{\tau}\right)\frac{1}{\epsilon} \\ &\quad - \frac{55}{4}\zeta_4 + \left(\zeta_2\zeta_3 - 41\zeta_5 + \zeta_4\ln\frac{-t}{\tau}\right)\epsilon \\ &\quad - \left(\frac{95}{2}\zeta_3^2 + \frac{1695}{8}\zeta_6 + (18\zeta_2\zeta_3 + 42\zeta_5)\ln\frac{-t}{\tau}\right)\epsilon^2 + \mathcal{O}(\epsilon^3) \\ &= \frac{1}{2}\left[C^{(1)}(t,\tau)\right]^2 + \frac{\zeta_2}{\epsilon^2} + \left(\zeta_3 + \zeta_2\ln\frac{-t}{\tau}\right)\frac{1}{\epsilon} \\ &\quad + \left(\zeta_3\ln\frac{-t}{\tau} - 19\zeta_4\right) + \left(4\zeta_4\ln\frac{-t}{\tau} - 2\zeta_2\zeta_3 - 39\zeta_5\right)\epsilon \\ &\quad - \left(48\zeta_3^2 + \frac{1773}{8}\zeta_6 + (18\zeta_2\zeta_3 + 41\zeta_5)\ln\frac{-t}{\tau}\right)\epsilon^2 + \mathcal{O}(\epsilon^3)\,. \end{aligned} \tag{4.8}$$

The three-loop coefficient is given by

$$\begin{aligned} m_4^{(3)} &= \frac{1}{3!}\left(\bar\alpha^{(1)}(t)\right)^3 L^3 + \bar\alpha^{(1)}(t)\left(\bar\alpha^{(2)}(t) + \bar C^{(1)}(t,\tau)\,\bar\alpha^{(1)}(t)\right)L^2 \\ &\quad + \left[\bar\alpha^{(3)}(t) + 2\,\bar\alpha^{(2)}(t)\,\bar C^{(1)}(t,\tau) + \bar\alpha^{(1)}(t)\left(2\,\bar C^{(2)}(t,\tau) + \left(\bar C^{(1)}(t,\tau)\right)^2\right)\right]L \\ &\quad + 2\,\bar C^{(3)}(t,\tau) + 2\,\bar C^{(2)}(t,\tau)\,\bar C^{(1)}(t,\tau) \\ &= m_4^{(2)} m_4^{(1)} - \frac{1}{3}\left(m_4^{(1)}\right)^3 \\ &\quad + \bar\alpha^{(3)}(t)L + 2\,\bar C^{(3)}(t,\tau) - 2\,\bar C^{(2)}(t,\tau)\,\bar C^{(1)}(t,\tau) + \frac{2}{3}\left(\bar C^{(1)}(t,\tau)\right)^3\,. \end{aligned} \tag{4.9}$$

In MSYM, the three-loop trajectory, $\alpha^{(3)}$, has been evaluated in ref. [26, 15, 37, 38] through $\mathcal{O}(\epsilon^0)$,

$$\alpha^{(3)} = \frac{44\zeta_4}{3\epsilon} + \frac{40}{3}\zeta_2\zeta_3 + 16\zeta_5 + \mathcal{O}(\epsilon)\,. \tag{4.10}$$

The three-loop coefficient function has been evaluated in ref. [26] through $\mathcal{O}(\epsilon^0)$ using

JHEP12(2008)097

knowledge of $m_4^{(1)}$ to $\mathcal{O}(\epsilon^4)$, and $m_4^{(2)}$ to $\mathcal{O}(\epsilon^2)$,

$$\begin{aligned} C^{(3)}(t,\tau) &= -\frac{4}{3\epsilon^6} - \frac{2}{\epsilon^5}\ln\frac{-t}{\tau} + \left(4\zeta_2 - \ln^2\frac{-t}{\tau}\right)\frac{1}{\epsilon^4} \\ &\quad + \left(3\zeta_2\ln\frac{-t}{\tau} - \frac{1}{6}\ln^3\frac{-t}{\tau}\right)\frac{1}{\epsilon^3} + \left(\frac{217\zeta_4}{9} + \frac{\zeta_2}{2}\ln^2\frac{-t}{\tau} - \zeta_3\ln\frac{-t}{\tau}\right)\frac{1}{\epsilon^2} \\ &\quad + \left(-\frac{22}{9}\zeta_2\zeta_3 + \frac{224}{3}\zeta_5 - \frac{\zeta_3}{2}\ln^2\frac{-t}{\tau} + \frac{71}{12}\zeta_4\ln\frac{-t}{\tau}\right)\frac{1}{\epsilon} \\ &\quad + \frac{796}{9}\zeta_3^2 + \frac{211861}{432}\zeta_6 - \frac{5}{2}\zeta_4\ln^2\frac{-t}{\tau} + \left(115\zeta_5 + \frac{97}{3}\zeta_2\zeta_3\right)\ln\frac{-t}{\tau} + \mathcal{O}(\epsilon) \\ &= C^{(2)}(t,\tau)\,C^{(1)}(t,\tau) - \frac{1}{3}\left[C^{(1)}(t,\tau)\right]^3 \\ &\quad - \frac{44}{9}\frac{\zeta_4}{\epsilon^2} - \left(\frac{40}{9}\zeta_2\zeta_3 + \frac{16}{3}\zeta_5 + \frac{22}{3}\zeta_4\ln\frac{-t}{\tau}\right)\frac{1}{\epsilon} \\ &\quad + \frac{3982}{27}\zeta_6 - \frac{68}{9}\zeta_3^2 - \left(8\zeta_5 + \frac{20}{3}\zeta_2\zeta_3\right)\ln\frac{-t}{\tau} + \mathcal{O}(\epsilon) \end{aligned} \tag{4.11}$$

It is straightforward to obtain the four-point amplitude in the physical region, $s \gg -t$, by continuing eqs. (4.3), (4.6) and (4.9) through the prescription $\ln(-s) = \ln(s) - i\pi$, for $s > 0$.

4.2 The five-point amplitude in multi-Regge kinematics

For the five-point amplitude, $g_1\, g_2 \to g_3\, g_4\, g_5$, the high-energy prescription (3.4) becomes

$$m_5 = s\,[g\,C(p_2,p_3,\tau)]\,\frac{1}{t_2}\left(\frac{-s_2}{\tau}\right)^{\alpha(t_2)}[g\,V(q_2,q_1,\kappa,\tau)]\,\frac{1}{t_1}\left(\frac{-s_1}{\tau}\right)^{\alpha(t_1)}[g\,C(p_1,p_5,\tau)]\;, \tag{4.12}$$

where $p_4 = q_2 - q_1$, and with the invariants labelled as in section 2, i.e. $t_1 = s_{51}$, $t_2 = s_{23}$, $s_1 = s_{45}$ and $s_2 = s_{34}$. In order for the amplitude m_5 to be real, eq. (4.12) is taken in the region where all the invariants are negative. Thus, the multi-Regge kinematics (3.6) become,

$$-s \gg -s_1, -s_2 \gg -t_1, -t_2\,. \tag{4.13}$$

Then the mass-shell condition (3.7) for the intermediate gluon 4 is

$$-\kappa = \frac{(-s_1)\,(-s_2)}{-s}\,, \tag{4.14}$$

where $\kappa = -|p_{4\perp}|^2$. In the expansion of eq. (3.12) for $n = 5$, the knowledge of the l-loop five-point amplitude in the multi-Regge kinematics (4.13), together with the l-loop trajectory $\alpha^{(l)}$ and coefficient function $C^{(l)}$, allows one to derive the Lipatov vertex to the same accuracy. The one-loop coefficient is

$$m_5^{(1)} = \bar\alpha^{(1)}(t_1)L_1 + \bar\alpha^{(1)}(t_2)L_2 + \bar C^{(1)}(t_1,\tau) + \bar C^{(1)}(t_2,\tau) + \bar V^{(1)}(t_1,t_2,\kappa,\tau)\,. \tag{4.15}$$

where $L_i = \ln(-s_i/\tau)$ and $i = 1,2$. Then subtracting the one-loop trajectory (4.4) and coefficient function (4.5) from the one-loop five-point amplitude, we can derive the one-loop Lipatov vertex. That will explicitly be done in a forthcoming publication.

In the expansion of eq. (3.12) for $n = 5$, the two-loop coefficient is

$$\begin{aligned} m_5^{(2)} &= \frac{1}{2}\left(m_5^{(1)}\right)^2 + \bar\alpha^{(2)}(t_1)L_1 + \bar\alpha^{(2)}(t_2)L_2 \\ &+\bar C^{(2)}(t_1,\tau) + \bar V^{(2)}(t_1,t_2,\kappa,\tau) + \bar C^{(2)}(t_2,\tau) \\ &-\frac{1}{2}\left(\bar C^{(1)}(t_1,\tau)\right)^2 - \frac{1}{2}\left(\bar V^{(1)}(t_1,t_2,\kappa,\tau)\right)^2 - \frac{1}{2}\left(\bar C^{(1)}(t_2,\tau)\right)^2 , \end{aligned} \tag{4.16}$$

where $m_5^{(1)}$, $\bar C^{(1)}(t,\tau)$ and $\bar V^{(1)}(t_1,t_2,\kappa,\tau)$ must be known to $\mathcal{O}(\epsilon^2)$. Similarly, the three-loop coefficient is

$$\begin{aligned} m_5^{(3)} &= m_5^{(2)}m_5^{(1)} - \frac{1}{3}\left(m_5^{(1)}\right)^3 + \bar\alpha^{(3)}(t_1)L_1 + \bar\alpha^{(3)}(t_2)L_2 \\ &+\bar C^{(3)}(t_1,\tau) + \bar V^{(3)}(t_1,t_2,\kappa,\tau) + \bar C^{(3)}(t_2,\tau) \\ &-\bar C^{(2)}(t_1,\tau)\,\bar C^{(1)}(t_1,\tau) - \bar V^{(2)}(t_1,t_2,\kappa,\tau)\bar V^{(1)}(t_1,t_2,\kappa,\tau) - \bar C^{(2)}(t_2,\tau)\,\bar C^{(1)}(t_2,\tau) \\ &+\frac{1}{3}\left(\bar C^{(1)}(t_1,\tau)\right)^3 + \frac{1}{3}\left(\bar V^{(1)}(t_1,t_2,\kappa,\tau)\right)^3 + \frac{1}{3}\left(\bar C^{(1)}(t_2,\tau)\right)^3 . \end{aligned} \tag{4.17}$$

Here, to find $m_5^{(3)}$ to $\mathcal{O}(\epsilon^0)$, $m_5^{(1)}$, $\bar C^{(1)}(t,\tau)$ and $\bar V^{(1)}(t_1,t_2,\kappa,\tau)$ must be known to $\mathcal{O}(\epsilon^4)$ while $m_5^{(2)}$, $\bar C^{(2)}(t,\tau)$ and $\bar V^{(2)}(t_1,t_2,\kappa,\tau)$ must be known to $\mathcal{O}(\epsilon^2)$.

It is straightforward to obtain the amplitudes in the physical region where s, s_1, s_2 are positive and t_1, t_2 are negative, and where the multi-Regge kinematics are

$$s \gg s_1,\ s_2 \gg -t_1,\ -t_2\,. \tag{4.18}$$

and the mass-shell condition is

$$\kappa = \frac{s_1\, s_2}{s}\,, \tag{4.19}$$

by continuing eqs. (4.15) and (4.16) through the prescriptions $\ln(-s_j) = \ln(s_j) - i\pi$, for $s_j > 0$ and $j = 1, 2$ and $\ln(-\kappa) = \ln(\kappa) - i\pi$, for $\kappa > 0$, which implies eq. (3.14) for the Lipatov vertex.

4.3 The six-point amplitude in multi-Regge kinematics

For the six-gluon amplitude, $g_1\, g_2 \to g_3\, g_4\, g_5\, g_6$, the high-energy prescription (3.4) becomes

$$\begin{aligned} m_6 &= s\,[g\,C(p_2,p_3,\tau)]\,\frac{1}{t_3}\left(\frac{-s_3}{\tau}\right)^{\alpha(t_3)}[g\,V(q_2,q_3,\kappa_2,\tau)] \\ &\times\frac{1}{t_2}\left(\frac{-s_2}{\tau}\right)^{\alpha(t_2)}[g\,V(q_1,q_2,\kappa_1,\tau)]\,\frac{1}{t_1}\left(\frac{-s_1}{\tau}\right)^{\alpha(t_1)}[g\,C(p_1,p_6,\tau)]\,. \end{aligned} \tag{4.20}$$

with $t_1 = s_{61}$, $t_2 = s_{234}$ and $t_3 = s_{23}$, $s_1 = s_{56}$, $s_2 = s_{45}$ and $s_3 = s_{34}$. In order for m_6 to be real, we take eq. (4.20) in the unphysical region where the invariants $s, s_1, s_2, s_3, t_1, t_2, t_3$ are all negative, where the multi-Regge kinematics are,

$$-s \gg -s_1, -s_2, -s_3 \gg -t_1, -t_2, -t_3\,, \tag{4.21}$$

and the on-shell conditions (3.7) are,

$$-\kappa_1 = \frac{(-s_1)\,(-s_2)}{(-s_{456})}\,, \qquad -\kappa_2 = \frac{(-s_2)\,(-s_3)}{(-s_{345})}\,, \tag{4.22}$$

with $\kappa_1 = -|p_{5\perp}|^2$ and $\kappa_2 = -|p_{4\perp}|^2$. Because in eq. (4.20) no new vertex or coefficient function occurs with respect to eq. (4.12), in the expansion of eq. (3.12) for $n = 6$, the knowledge of the l-loop trajectory $\alpha^{(l)}$, the coefficient function $C^{(l)}$, and the Lipatov vertex $V^{(l)}$ allow one to derive the l-loop six-point amplitude in the multi-Regge kinematics. The one-loop coefficient is

$$\begin{aligned} m_6^{(1)} &= \bar{\alpha}^{(1)}(t_1)L_1 + \bar{\alpha}^{(1)}(t_2)L_2 + \bar{\alpha}^{(1)}(t_3)L_3 \\ &+\bar{C}^{(1)}(t_1,\tau) + \bar{C}^{(1)}(t_3,\tau) + \bar{V}^{(1)}(t_1,t_2,\kappa_1,\tau) + \bar{V}^{(1)}(t_2,t_3,\kappa_2,\tau). \end{aligned} \tag{4.23}$$

with $L_i = \ln(-s_i/\tau)$ and $i = 1,2,3$. The two-loop coefficient is

$$\begin{aligned} m_6^{(2)} &= \frac{1}{2}\left(m_6^{(1)}\right)^2 + \bar{\alpha}^{(2)}(t_1)L_1 + \bar{\alpha}^{(2)}(t_2)L_2 + \bar{\alpha}^{(2)}(t_3)L_3 \\ &+\bar{C}^{(2)}(t_1,\tau) + \bar{C}^{(2)}(t_3,\tau) + \bar{V}^{(2)}(t_1,t_2,\kappa_1,\tau) + \bar{V}^{(2)}(t_2,t_3,\kappa_2,\tau) \\ &-\frac{1}{2}\left(\bar{C}^{(1)}(t_1,\tau)\right)^2 - \frac{1}{2}\left(\bar{C}^{(1)}(t_3,\tau)\right)^2 \\ &-\frac{1}{2}\left(\bar{V}^{(1)}(t_1,t_2,\kappa_1,\tau)\right)^2 - \frac{1}{2}\left(\bar{V}^{(1)}(t_2,t_3,\kappa_2,\tau)\right)^2 , \end{aligned} \tag{4.24}$$

where $m_6^{(1)}$, $\bar{C}^{(1)}(t,\tau)$ and $\bar{V}^{(1)}(t_1,t_2,\kappa,\tau)$ must be known to $\mathcal{O}(\epsilon^2)$. Similarly, the three-loop coefficient is

$$\begin{aligned} m_6^{(3)} &= m_6^{(2)}m_6^{(1)} - \frac{1}{3}\left(m_6^{(1)}\right)^3 + \bar{\alpha}^{(3)}(t_1)L_1 + \bar{\alpha}^{(3)}(t_3)L_2 \\ &+\bar{C}^{(3)}(t_1,\tau) + \bar{V}^{(3)}(t_1,t_2,\kappa_1,\tau) + \bar{V}^{(3)}(t_2,t_3,\kappa_2,\tau) + \bar{C}^{(3)}(t_3,\tau) \\ &-\bar{C}^{(2)}(t_1,\tau)\,\bar{C}^{(1)}(t_1,\tau) - \bar{V}^{(2)}(t_1,t_2,\kappa_1,\tau)\bar{V}^{(1)}(t_1,t_2,\kappa_1,\tau) \\ &-\bar{V}^{(2)}(t_2,t_3,\kappa_2,\tau)\bar{V}^{(1)}(t_2,t_3,\kappa_2,\tau) - \bar{C}^{(2)}(t_3,\tau)\,\bar{C}^{(1)}(t_3,\tau) \\ &+\frac{1}{3}\left(\bar{C}^{(1)}(t_1,\tau)\right)^3 + \frac{1}{3}\left(\bar{C}^{(1)}(t_3,\tau)\right)^3 \\ &+\frac{1}{3}\left(\bar{V}^{(1)}(t_1,t_2,\kappa_1,\tau)\right)^3 + \frac{1}{3}\left(\bar{V}^{(1)}(t_2,t_3,\kappa_2,\tau)\right)^3 . \end{aligned} \tag{4.25}$$

Here, $m_6^{(1)}$, $\bar{C}^{(1)}(t,\tau)$ and $\bar{V}^{(1)}(t_1,t_2,\kappa,\tau)$ are needed to $\mathcal{O}(\epsilon^4)$ while $m_6^{(2)}$, $\bar{C}^{(2)}(t,\tau)$ and $\bar{V}^{(2)}(t_1,t_2,\kappa,\tau)$ must be known to $\mathcal{O}(\epsilon^2)$.

It is straightforward to obtain the amplitudes in the physical region where s, s_1, s_2, s_3 are positive and t_1, t_2, t_3 are negative, where the multi-Regge kinematics are

$$s \gg s_1, s_2, s_3 \gg -t_1, -t_2, -t_3 \,, \tag{4.26}$$

and the mass-shell conditions for gluons 4 and 5, emitted along the t channel, are

$$\kappa_1 = \frac{s_1\, s_2}{s_{456}}\,, \qquad \kappa_2 = \frac{s_2\, s_3}{s_{345}}\,, \tag{4.27}$$

by continuing eqs. (4.23) and (4.24) through the prescriptions $\ln(-s_j) = \ln(s_j) - i\pi$, for $s_j > 0$ with $j = 1,2,3$, and $\ln(-\kappa_i) = \ln(\kappa_i) - i\pi$, for $\kappa_i > 0$, with $i = 1,2$.

JHEP12(2008)097

5. The Bern-Dixon-Smirnov ansatz in multi-Regge kinematics

The BDS ansatz prescribes that the n-gluon MHV amplitude be written as,

$$\begin{aligned} m_n &= m_n^{(0)} \left[1 + \sum_{L=1}^{\infty} a^L M_n^{(L)}(\epsilon) \right] \\ &= m_n^{(0)} \exp \left[\sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + Const^{(l)} + E_n^{(l)}(\epsilon) \right) \right], \end{aligned} \tag{5.1}$$

where

$$a = \frac{2g^2 N}{(4\pi)^{2-\epsilon}} e^{-\gamma\epsilon} \tag{5.2}$$

is the 't-Hooft gauge coupling, and with

$$f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon f_1^{(l)} + \epsilon^2 f_2^{(l)}, \tag{5.3}$$

where $f^{(1)}(\epsilon) = 1$, and $f_0^{(l)}$ is proportional to the l-loop cusp anomalous dimension [4], $\hat{\gamma}_K^{(l)} = 4 f_0^{(l)}$, which has been conjectured to all orders of a [5] and computed to $\mathcal{O}(a^4)$ [39, 40], and $f_1^{(l)}$ is related to the soft anomalous dimension [6, 7], $\mathcal{G}_0^{(l)} = 2 f_1^{(l)}/l$, and is known to $\mathcal{O}(a^3)$ [1]. In eq. (5.1), $Const^{(l)}$ are constants, and $E_n^{(l)}(\epsilon)$ are $\mathcal{O}(\epsilon)$ contributions, with $Const^{(1)} = 0$ and $E_n^{(1)}(\epsilon) = 0$, and $M_n^{(L)}(\epsilon)$ is the L-loop colour-stripped amplitude rescaled by the tree amplitude. In the convention and notation of eq. (3.12), the rescaled coupling (3.9) is related to a by,

$$a = 2G(\epsilon)\bar{g}^2 \tag{5.4}$$

with

$$G(\epsilon) = \frac{e^{-\gamma\epsilon}\,\Gamma(1-2\epsilon)}{\Gamma(1+\epsilon)\,\Gamma^2(1-\epsilon)} = 1 + \mathcal{O}(\epsilon^2). \tag{5.5}$$

Thus, the n-gluon amplitude is given by,

$$a^L M_n^{(L)}(\epsilon) = \left(\frac{a}{2G(\epsilon)} \right)^L m_n^{(L)}(\epsilon), \tag{5.6}$$

and the BDS ansatz (5.1) becomes

$$\begin{aligned} m_n &= m_n^{(0)} \left[1 + \sum_{L=1}^{\infty} \bar{g}^{2L}(t) m_n^{(L)}(\epsilon) \right] \\ &= m_n^{(0)} \exp \left[\sum_{l=1}^{\infty} \bar{g}^{2l}(t)\,(2G(\epsilon))^l \left(f^{(l)}(\epsilon) \frac{m_n^{(1)}(l\epsilon)}{2G(l\epsilon)} + Const^{(l)} + E_n^{(l)}(\epsilon) \right) \right]. \end{aligned} \tag{5.7}$$

5.1 Amplitudes with four or five gluons

Substituting the one-loop four-point amplitude (4.3) in eq. (5.7) and comparing with the expansion (3.12) for $n = 4$ of the high-energy prescription (4.1), we determine the Regge trajectory from the coefficient of the single logarithm [26],

$$\begin{aligned} \alpha^{(2)}(\epsilon) &= 2\, f^{(2)}(\epsilon)\, \alpha^{(1)}(2\epsilon) + \mathcal{O}(\epsilon), \\ \alpha^{(3)}(\epsilon) &= 4\, f^{(3)}(\epsilon)\, \alpha^{(1)}(3\epsilon) + \mathcal{O}(\epsilon), \end{aligned} \tag{5.8}$$

with $\alpha^{(1)}$ given in eq. (4.4), and in general

$$\alpha^{(l)}(\epsilon) = 2^{l-1}\, f^{(l)}(\epsilon)\, \alpha^{(1)}(l\epsilon) + \mathcal{O}(\epsilon)\,. \tag{5.9}$$

From eq. (5.9), we see that to $\mathcal{O}(\epsilon^0)$ only the first two terms of the $f^{(l)}(\epsilon)$ function (5.3) enter the evaluation of the Regge trajectory. Using the $f^{(2)}$ and $f^{(3)}$ functions [1],

$$\begin{aligned} f^{(2)}(\epsilon) &= -\zeta_2 - \zeta_3\epsilon - \zeta_4\epsilon^2\,, \\ f^{(3)}(\epsilon) &= \frac{11}{2}\zeta_4 + (6\zeta_5 + 5\zeta_2\zeta_3)\epsilon + (c_1\zeta_6 + c_2\zeta_3^2)\epsilon^2\,, \end{aligned} \tag{5.10}$$

we see that eq. (5.8) agrees with eqs. (4.7) and (4.10) to $\mathcal{O}(\epsilon^0)$. The constants c_1, c_2 are known only numerically [41], but they do not enter the evaluation of the Regge trajectory.

Eq. (5.7) implies the iterative structure of the two-loop n-gluon amplitude given in eq. (1.1), which we report here in our convention (5.6) for the coupling,

$$m_n^{(2)}(\epsilon) = \frac{1}{2}\left[m_n^{(1)}(\epsilon)\right]^2 + \frac{2\,G^2(\epsilon)}{G(2\epsilon)} f^{(2)}(\epsilon)\, m_n^{(1)}(2\epsilon) + 4\,Const^{(2)} + \mathcal{O}(\epsilon)\,, \tag{5.11}$$

with $Const^{(2)} = -\zeta_2^2/2$, and where the one-loop amplitude, $m_n^{(1)}(\epsilon)$, must be known to $\mathcal{O}(\epsilon^2)$. Eq. (5.11) has been shown to be correct for $n = 4$ [42] and $n = 5$ [2, 3] for general kinematics.

Using the iterative structure (5.11) for the four-point amplitude, it is possible to express the two-loop coefficient function in terms of the one-loop coefficient function. In fact, comparing eq. (5.11) with $n = 4$ to the two-loop factorization of the four-point amplitude in the multi-Regge kinematics (4.6), we find the following iterative structure

$$C^{(2)}(t,\tau,\epsilon) = \frac{1}{2}\left[C^{(1)}(t,\tau,\epsilon)\right]^2 + \frac{2\,G^2(\epsilon)}{G(2\epsilon)} f^{(2)}(\epsilon)\, C^{(1)}(t,\tau,2\epsilon) + 2\,Const^{(2)} + \mathcal{O}(\epsilon)\,, \tag{5.12}$$

where, to compute the two-loop coefficient function $C^{(2)}(t,\tau,\epsilon)$ to $\mathcal{O}(\epsilon^0)$, the one-loop coefficient function, $C^{(1)}(t,\tau,\epsilon)$, is needed to $\mathcal{O}(\epsilon^2)$. Eq. (5.12) agrees with eq. (4.8) to $\mathcal{O}(\epsilon^0)$.

Similarly, the iterative structure (5.11) for the five-point amplitude, means we can also express the two-loop Lipatov vertex in terms of the one-loop Lipatov vertex. Comparing eq. (5.11) with $n = 5$ to the two-loop factorization of the five-point amplitude (4.16), and using eqs. (5.8) and (5.12), we obtain

$$V^{(2)}(t_1,t_2,\kappa,\tau,\epsilon) = \frac{1}{2}\left[V^{(1)}(t_1,t_2,\kappa,\tau,\epsilon)\right]^2 + \frac{2\,G^2(\epsilon)}{G(2\epsilon)} f^{(2)}(\epsilon)\, V^{(1)}(t_1,t_2,\kappa,\tau,2\epsilon) + \mathcal{O}(\epsilon)\,, \tag{5.13}$$

where, to compute $V^{(2)}(t_1,t_2,\kappa,\tau,\epsilon)$ to $\mathcal{O}(\epsilon^0)$, $V^{(1)}(t_1,t_2,\kappa,\tau,\epsilon)$, must be known through $\mathcal{O}(\epsilon^2)$. Of course, eq. (5.11) with $n = 5$ requires the knowledge of the one-loop five-point amplitude, $m_5^{(1)}(\epsilon)$, through $\mathcal{O}(\epsilon^2)$,[4] but once $V^{(1)}$ is known through $\mathcal{O}(\epsilon^2)$, the two-loop Lipatov vertex can be determined by eq. (5.13) without knowing explicitly the two-loop

[4]We shall provide the details of $m_5^{(1)}(\epsilon)$ to that accuracy, in fact to all orders in ϵ, in a forthcoming publication [43].

JHEP12(2008)097

five-point amplitude. In fact, once evaluated, $V^{(2)}$ can be used, together with $C^{(2)}$ and $\alpha^{(2)}$, in eq. (4.16) to determine the two-loop five-point amplitude in the multi-Regge kinematics.

The iterative structure of the three-loop n-gluon amplitude is,

$$m_n^{(3)}(\epsilon) = m_n^{(2)}(\epsilon)\, m_n^{(1)}(\epsilon) - \frac{1}{3}\left[m_n^{(1)}(\epsilon)\right]^3 + \frac{4\,G^3(\epsilon)}{G(3\epsilon)} f^{(3)}(\epsilon)\, m_n^{(1)}(3\epsilon) + 8\,Const^{(3)} + \mathcal{O}(\epsilon)\,, \tag{5.14}$$

where $m_n^{(1)}(\epsilon)$ and $m_n^{(2)}(\epsilon)$ must be known to $\mathcal{O}(\epsilon^4)$ and $\mathcal{O}(\epsilon^2)$, respectively, and with

$$Const^{(3)} = \left(\frac{341}{216} + \frac{2}{9}c_1\right)\zeta_6 + \left(-\frac{17}{9} + \frac{2}{9}c_2\right)\zeta_3^2\,. \tag{5.15}$$

Eq. (5.14) has been shown to be correct for $n = 4$ [1].

Comparing eq. (5.14) with $n = 4$ to the three-loop factorisation of the four-point amplitude in the multi-Regge kinematics (4.9), we obtain the three-loop iteration of the coefficient function,

$$\begin{aligned} C^{(3)}(t,\tau,\epsilon) &= C^{(2)}(t,\tau,\epsilon)\, C^{(1)}(t,\tau,\epsilon) - \frac{1}{3}\left[C^{(1)}(t,\tau,\epsilon)\right]^3 \\ &\quad + \frac{4\,G^3(\epsilon)}{G(3\epsilon)} f^{(3)}(\epsilon)\, C^{(1)}(t,\tau,3\epsilon) + 4\,Const^{(3)} + \mathcal{O}(\epsilon)\,. \end{aligned} \tag{5.16}$$

The constants c_1, c_2 cancel when eqs. (5.10) and (5.15) are used in eqs. (5.14) and (5.16). Using the two-loop coefficient function to $\mathcal{O}(\epsilon^2)$ (4.8), and the one-loop coefficient function to $\mathcal{O}(\epsilon^4)$ (4.5), we see that eq. (5.16) is in agreement with eq. (4.11) to $\mathcal{O}(\epsilon^0)$.

Comparing eq. (5.14) with $n = 5$ to the three-loop factorisation of the five-point amplitude (4.17), we obtain the three-loop iteration of the Lipatov vertex,

$$\begin{aligned} V^{(3)}(t_1,t_2,\kappa,\tau,\epsilon) &= V^{(2)}(t_1,t_2,\kappa,\tau,\epsilon) V^{(1)}(t_1,t_2,\kappa,\tau,\epsilon) - \frac{1}{3}\left[V^{(1)}(t_1,t_2,\kappa,\tau,\epsilon)\right]^3 \\ &\quad + \frac{4\,G^3(\epsilon)}{G(3\epsilon)} f^{(3)}(\epsilon)\, V^{(1)}(t_1,t_2,\kappa,\tau,3\epsilon) + \mathcal{O}(\epsilon)\,. \end{aligned} \tag{5.17}$$

5.2 Amplitudes with six or more gluons

In the two-loop expansion of the six-point amplitude (4.24), no new vertices or coefficient functions occur. Thus, using the explicit expressions of $V^{(2)}$, $C^{(2)}$ and $\alpha^{(2)}$ in eq. (4.24), one can assemble the two-loop six-point amplitude in the multi-Regge kinematics. However, even without knowing the explicit expression of the two-loop Lipatov vertex (5.13), it is easy to see by substitution that the iterative structure of eqs. (5.8), (5.12) and (5.13) ensures that the six-point amplitude (4.24) fulfils the two-loop iterative formula (5.11) for $n = 6$. Furthermore, the expression has the correct analytic properties in the physical region where s, s_1, s_2, s_3 are positive and t_1, t_2, t_3 are negative.

Because no new vertices or coefficient functions occur in the two-loop expansion of eq. (3.4) even for $n = 7$ or higher, we conclude that the two-loop expansion of eq. (3.4) fulfils the two-loop iterative formula (5.11), and thus the BDS ansatz, for any n. Thus, the multi-Regge kinematics are not able to resolve the BDS-ansatz discrepancy, i.e. the quantity $R_n^{(2)}$ (1.2) vanishes in the multi-Regge kinematics, for any n.

The same arguments can be repeated for three-loop case: in the three-loop expansion of the six-point amplitude (4.25) no new vertices or coefficient functions occur. Thus, using the explicit expressions of $V^{(3)}$, $C^{(3)}$ and $\alpha^{(3)}$ in eq. (4.25) one can assemble the three-loop six-point amplitude in the multi-Regge kinematics. However, even without knowing the explicit expression of the three-loop Lipatov vertex (5.17), it is easy to see by substitution that the iterative structure of eqs. (5.8), (5.16) and (5.17) ensures that the six-point amplitude (4.25) fulfils the three-loop iterative formula (5.14) for $n = 6$. Because no new vertices or coefficient functions occur in the three-loop expansion of eq. (3.4) for $n = 7$ or higher, the three-loop expansion of eq. (3.4) fulfils the three-loop iterative formula (5.14), and thus the BDS ansatz, for any n. Thus, also the quantity $R_n^{(3)}$ (1.2) vanishes in the multi-Regge kinematics, for any n. Clearly, the same thing is to occur with the iterative structure of the l-loop n-gluon amplitude for $l \geq 4$. We conclude that $R_n^{(l)}$ vanishes in the multi-Regge kinematics for any l and n. The l-loop n-gluon amplitudes in the multi-Regge kinematics are in complete agreement with the BDS ansatz, therefore they are not able to resolve the violations of the ansatz for $n \geq 6$.

In ref. [11, 44] it was argued that the remainder function (1.2) for $n = 6$ is a function of the three conformal cross-ratios

$$u_1 = \frac{s_{12}\, s_{45}}{s_{345}\, s_{456}}\,, \qquad u_2 = \frac{s_{23}\, s_{56}}{s_{234}\, s_{456}}\,, \qquad u_3 = \frac{s_{34}\, s_{61}}{s_{234}\, s_{345}}\,. \tag{5.18}$$

Using the notation of section 2 and the results of section B.1, we note that in the multi-Regge kinematics (4.26) the conformal invariants (5.18) become [45, 46]

$$u_1 \simeq 1\,, \qquad u_2 = \frac{t_3 \kappa_1}{t_2 s_2} \simeq \mathcal{O}\left(\frac{t}{s}\right)\,, \qquad u_3 = \frac{t_1 \kappa_2}{t_2 s_2} \simeq \mathcal{O}\left(\frac{t}{s}\right)\,, \tag{5.19}$$

thus u_1 is close to 1, while u_2 and u_3 are very small and are in fact sub-leading in the multi-Regge kinematics.

6. Proof of BDS ansatz in multi-Regge kinematics

In the previous section, we derived iterative relations for the three building blocks that occur in the multi-Regge factorisation of gluonic amplitudes, the Regge trajectory, the coefficient functions and the Lipatov vertex. We argued that the high-energy prescription implied that the six-gluon amplitude also satisfies the BDS ansatz (in the restricted kinematics where the high energy prescription is valid). In this section, we are going to prove that the BDS ansatz is fully consistent with multi-Regge factorisation. In particular, we show that, if BDS holds true for four- and five-point amplitudes, then it also holds true for any n-gluon amplitude (in multi-Regge kinematics).

We start by deriving exponentiated forms for the coefficient functions and the Lipatov vertex. If the BDS ansatz holds true for the four-point amplitude, then we can immediately insert the tree- and one-loop four-gluon amplitudes in multi-Regge kinematics

$$\begin{aligned} m_4^{(0)} =& g^2 C^{(0)}(p_2, p_3)\, \frac{s}{t}\, C^{(0)}(p_1, p_4), \\ m_4^{(1)}(l\epsilon) =& 2\bar{C}^{(1)}(t, \tau, l\epsilon) + \bar{\alpha}^{(1)}(t, l\epsilon) \ln\left(\frac{-s}{\tau}\right), \end{aligned} \tag{6.1}$$

into eq. (5.7), such that

$$m_4 = g^2 C^{(0)}(p_2, p_3)\, \frac{s}{t}\, C^{(0)}(p_1, p_4) \left(\frac{-s}{\tau}\right)^{\sum_{l=1}^{\infty} \bar{g}^{2l}\, 2^{l-1} \frac{G^l(\epsilon)}{G(l\epsilon)} f^{(l)}(\epsilon)\, \bar{\alpha}^{(1)}(t,l\epsilon)}$$
$$\times \exp 2 \sum_{l=1}^{\infty} \bar{g}^{2l}\, 2^{l-1} G^l(\epsilon) \left(\frac{f^{(l)}(\epsilon)}{G(l\epsilon)}\, \bar{C}^{(1)}(t, \tau, l\epsilon) + Const^{(l)} + E_4^{(l)}(\epsilon) \right). \quad (6.2)$$

Comparing eq. (6.2) to the general form of the high energy prescription of eq. (4.1), we can easily identify the all-orders forms of the Regge trajectory

$$\alpha(t, \epsilon) = \sum_{l=1}^{\infty} \bar{g}^{2l}\, 2^{l-1} \frac{G^l(\epsilon)}{G(l\epsilon)}\, f^{(l)}(\epsilon)\, \bar{\alpha}^{(1)}(t, l\epsilon), \quad (6.3)$$

and the coefficient function,

$$C(p_i, p_j, \tau, \epsilon) = \quad (6.4)$$
$$C^{(0)}(p_i, p_j)\, \exp \sum_{l=1}^{\infty} \bar{g}^{2l}\, 2^{l-1} G^l(\epsilon) \left(\frac{f^{(l)}(\epsilon)}{G(l\epsilon)}\, \bar{C}^{(1)}(t, \tau, l\epsilon) + Const^{(l)} + E_4^{(l)}(\epsilon) \right),$$

where in the last equation $t = (p_i + p_j)^2$. Note that expanding eq. (6.3) and eq. (6.4) in the rescaled couplings reproduces the explicit forms for the two-, and three-loop iterative expressions given in eq. (5.8) and eqs. (5.12) and (5.16) respectively. Furthermore, eq. (6.3) is in agreement up to $\mathcal{O}(\epsilon)$ with eq. (5.9), which expresses the l-loop Regge trajectory in terms of the function $f^{(l)}$ appearing in the BDS ansatz.

We can now repeat the argument for m_5 and, by reusing eq. (6.3) and eq. (6.4), extract the corresponding formula for the Lipatov vertex,

$$m_5 = g^2\, C(p_2, p_3, \tau, \epsilon)\, \frac{s}{t_1\, t_2}\, V^{(0)}(q_2, q_1)\, C(p_1, p_5, \tau, \epsilon)$$
$$\times \left(\frac{-s_1}{\tau}\right)^{\alpha(t_1, \epsilon)} \left(\frac{-s_2}{\tau}\right)^{\alpha(t_2, \epsilon)}$$
$$\times \exp \sum_{l=1}^{\infty} \bar{g}^{2l}\, 2^l G^l(\epsilon) \left(\frac{f^{(l)}(\epsilon)}{2G(l\epsilon)}\, \bar{V}^{(1)}(t_2, t_1, \kappa_1, \tau, l\epsilon) + E_5^{(l)}(\epsilon) - E_4^{(l)}(\epsilon) \right). \quad (6.5)$$

Comparing with eq. (4.12), we find

$$V(q_2, q_1, \kappa, \epsilon) = V^{(0)}(q_2, q_1)$$
$$\times \exp \sum_{l=1}^{\infty} \bar{g}^{2l}\, 2^l G^l(\epsilon) \left(\frac{f^{(l)}(\epsilon)}{2G(l\epsilon)}\, \bar{V}^{(1)}(t_2, t_1, \kappa_1, \tau, l\epsilon) + E_5^{(l)}(\epsilon) - E_4^{(l)}(\epsilon) \right). \quad (6.6)$$

As before, expanding eq. (6.6) in the rescaled couplings reproduces the explicit forms for the two-, and three-loop iterative expressions given in eqs. (5.13) and (5.17).

We now turn to the generic case. Consider an n-gluon amplitude in multi-Regge kinematics which satisfies eq. (7.2). Inserting the exponentiated expressions for the Regge

JHEP12(2008)097

trajectory eq. (6.3), the coefficient functions eq. (6.4) and the Lipatov vertex eq. (6.6), we find

$$m_n = m_n^{(0)} \exp \sum_{l=1}^{\infty} \bar{g}^{2l}\, 2^l G^l(\epsilon) \Bigg[\frac{f^{(l)}(\epsilon)}{2G(l\epsilon)} \Bigg(\bar{C}^{(1)}(t_1, \tau, l\epsilon) + \bar{C}^{(1)}(t_{n-3}, \tau, l\epsilon) + \sum_{k=1}^{n-3} \bar{\alpha}^{(1)}(t_k, l\epsilon) \ln\left(\frac{-s_k}{\tau}\right) + \sum_{k=1}^{n-4} \bar{V}^{(1)}(t_{k+1}, t_k, \kappa_k, \tau, l\epsilon) \Bigg) + Const^{(l)} + E_4^{(l)}(\epsilon) + (n-4)\big(E_5^{(l)}(\epsilon) - E_4^{(l)}(\epsilon)\big) \Bigg]. \quad (6.7)$$

The expression inside the brackets can now be easily identified as the one-loop amplitude in multi-Regge kinematics,

$$m_n^{(1)}(l\epsilon) = \bar{C}^{(1)}(t_1, \tau, l\epsilon) + \bar{C}^{(1)}(t_{n-3}, \tau, l\epsilon) + \sum_{k=1}^{n-3} \bar{\alpha}^{(1)}(t_k, l\epsilon) \ln\left(\frac{-s_k}{\tau}\right) + \sum_{k=1}^{n-4} \bar{V}^{(1)}(t_{k+1}, t_k, \kappa_k, \tau, l\epsilon), \quad (6.8)$$

and so we recover

$$m_n = m_n^{(0)} \exp \sum_{l=1}^{\infty} \bar{g}^{2l}\, 2^l G^l(\epsilon) \left(\frac{f^{(l)}(\epsilon)}{2G(l\epsilon)} m_n^{(1)}(l\epsilon) + Const^{(l)} + \mathcal{O}(\epsilon) \right), \quad (6.9)$$

i.e. m_n satisfies the BDS ansatz up to $\mathcal{O}(\epsilon)$.

7. Quasi-multi-Regge kinematics

7.1 Amplitudes in the quasi-multi-Regge kinematics with a pair at either end of the ladder

It is possible to define a high-energy prescription for more general, i.e. less restrictive, multi-Regge kinematics, such as the quasi-multi-Regge kinematics where all gluons are strongly ordered in rapidity, except for a pair of gluons, either at the top or at the bottom of the ladder as shown schematically in figure 2(a). For example,

$$y_3 \simeq y_4 \gg \cdots \gg y_n; \qquad |p_{3\perp}| \simeq |p_{4\perp}| \ldots \simeq |p_{n\perp}|\,, \quad (7.1)$$

for which the Mandelstam invariants are given in section D.1. We conjecture that in the quasi-multi-Regge kinematics of eq. (7.1) a generic colour-stripped l-loop n-gluon amplitude will have the factorised form,

$$m_n(1,2,\ldots,n) = s\left[g^2 A(p_2,p_3,p_4)\right] \frac{1}{t_{n-4}} \left(\frac{-s_{n-4}}{\tau}\right)^{\alpha(t_{n-4})} \left[g\,V(q_{n-4}, q_{n-5}, \kappa_{n-5})\right] \cdots \times \frac{1}{t_2}\left(\frac{-s_2}{\tau}\right)^{\alpha(t_2)} \left[g\,V(q_2,q_1,\kappa_1)\right] \frac{1}{t_1}\left(\frac{-s_1}{\tau}\right)^{\alpha(t_1)} \left[g\,C(p_1,p_n)\right]\,, \quad (7.2)$$

where we suppressed the dependence of the coefficient functions and the Lipatov vertices on the reggeisation scale τ. s_{n-4} can be chosen to be either s_{35} or s_{45}, the difference between the two being of the order of s_{34}, thus sub-leading with respect to s. In order for m_n to be real, one can take the invariants s, $s_1, \ldots, s_{n-4}$, $t_1, \ldots, t_{n-4}$, defined as in section 2, and s_{34} all negative. Then the kinematics imply

$$-s \gg -s_1, -s_2, \ldots, -s_{n-4} \gg -s_{34}, -t_1, -t_2, \ldots, -t_{n-4}\,. \tag{7.3}$$

The limit of multi-Regge kinematics (3.6) is where s_{34} becomes as large as any s_i-type invariant.

In eq. (7.2), the coefficient function C and Lipatov vertex V are exactly the same as in eq. (3.4). However a new coefficient function, $A(p_2, p_3, p_4)$, is needed to describe the production of two gluons at one end of the ladder. The tree approximation, $A^{(0)}(p_2, p_3, p_4)$, was computed in ref. [47, 48]. A can be expanded in the rescaled coupling, just as in eqs. (3.10) and (3.11),

$$A(p_2, p_3, p_4, \tau) = A^{(0)}(p_2, p_3, p_4)\left(1 + \bar{g}^2 \bar{A}^{(1)}(t, s_{34}, \tau) + \bar{g}^4 \bar{A}^{(2)}(t, s_{34}, \tau) + \mathcal{O}(\bar{g}^6)\right)\,. \tag{7.4}$$

For $n = 5$, eq. (7.2) reduces to

$$m_5(1, 2, 3, 4, 5) = s\left[g^2 A(p_2, p_3, p_4, \tau)\right] \frac{1}{t} \left(\frac{-s_1}{\tau}\right)^{\alpha(t)} \left[g\, C(p_1, p_5, \tau)\right]\,, \tag{7.5}$$

with $q = p_1 + p_5 = -(p_2 + p_3 + p_4)$, $t = q^2$ and $s = s_{12}$. Expanding eq. (7.5) as in eq. (3.12), we obtain, at one-, two- and three-loop accuracy,

$$\begin{aligned}
m_5^{(1)} &= \bar{\alpha}^{(1)}(t)L + \bar{C}^{(1)}(t, \tau) + \bar{A}^{(1)}(t, s_{34}, \tau)\,, & (7.6)\\
m_5^{(2)} &= \frac{1}{2}\left(m_5^{(1)}\right)^2 + \bar{\alpha}^{(2)}(t)L \\
&\quad + \bar{C}^{(2)}(t, \tau) + \bar{A}^{(2)}(t, s_{34}, \tau) - \frac{1}{2}\left(\bar{C}^{(1)}(t, \tau)\right)^2 - \frac{1}{2}\left(\bar{A}^{(1)}(t, s_{34}, \tau)\right)^2\,, & (7.7)\\
m_5^{(3)} &= m_5^{(2)}\, m_5^{(1)} - \frac{1}{3}\left(m_5^{(1)}\right)^3 + \bar{\alpha}^{(3)}(t)L \\
&\quad + \bar{C}^{(3)}(t, \tau) + \bar{A}^{(3)}(t, s_{34}, \tau) - \bar{C}^{(2)}(t, \tau)\bar{C}^{(1)}(t, \tau) - \bar{A}^{(2)}(t, s_{34}, \tau)\bar{A}^{(1)}(t, s_{34}, \tau) \\
&\quad + \frac{1}{3}\left(\bar{C}^{(1)}(t, \tau)\right)^3 + \frac{1}{3}\left(\bar{A}^{(1)}(t, s_{34}, \tau)\right)^3\,, & (7.8)
\end{aligned}$$

with $L = \ln(-s_1/\tau)$, and where $m_5^{(1)}$ is needed to $\mathcal{O}(\epsilon^2)$ in eq. (7.7), and $m_5^{(1)}$ and $m_5^{(2)}$ to $\mathcal{O}(\epsilon^4)$ and $\mathcal{O}(\epsilon^2)$ respectively in eq. (7.8). The coefficient functions $\bar{C}$ were already evaluated in section 4.1. Therefore, knowledge of the five-point amplitude at a given loop accuracy in the quasi-multi-Regge kinematics (7.1) allows one to find the coefficient function A at the same loop accuracy. Furthermore, combining the iterative formula (5.11) for $n = 5$ with the high-energy prescription (7.2), one obtains an iterative formula for the coefficient function A,

$$A^{(2)}(t, s_{34}, \tau, \epsilon) = \frac{1}{2}\left[A^{(1)}(t, s_{34}, \tau, \epsilon)\right]^2 + \frac{2\, G^2(\epsilon)}{G(2\epsilon)} f^{(2)}(\epsilon)\, A^{(1)}(t, s_{34}, \tau, 2\epsilon) + 2\, Const^{(2)} + \mathcal{O}(\epsilon)\,, \tag{7.9}$$

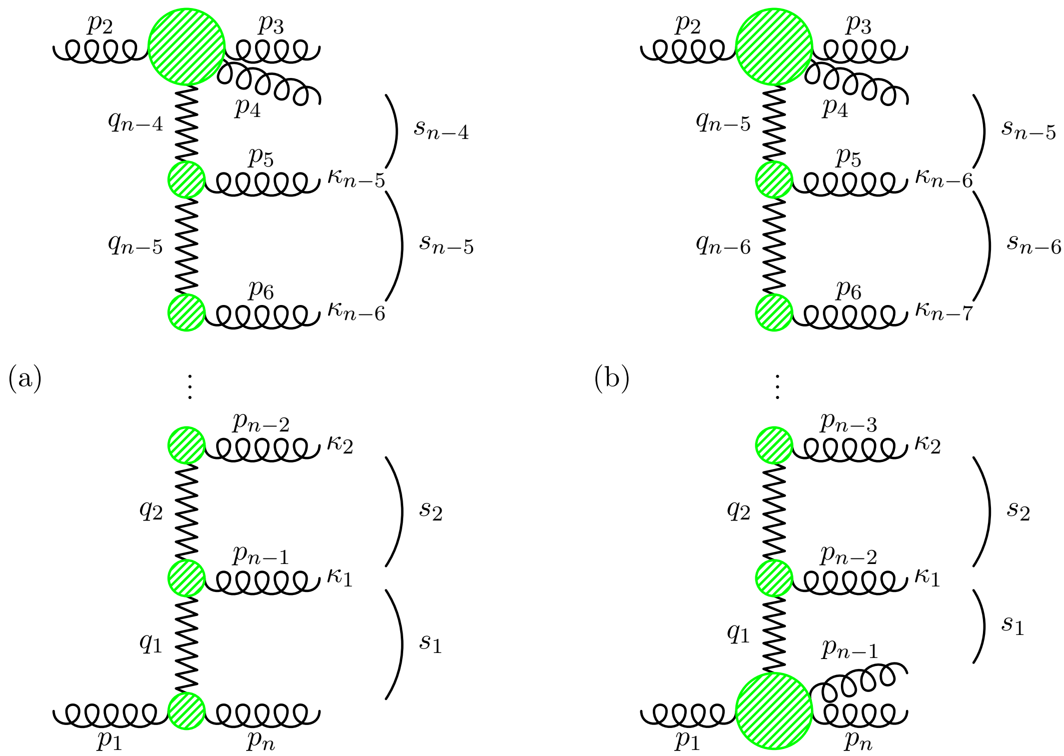

Figure 2: Amplitudes in the quasi-multi-Regge kinematics of (a) a pair at either end of the ladder and (b) two pairs, one at each end of the ladder.

where the one-loop coefficient function, $A^{(1)}(\epsilon)$, is needed to $\mathcal{O}(\epsilon^2)$. Similarly, it is straight forward to derive from eq. (7.8) an iterative formula at the three-loop coefficient function

$$\begin{aligned} A^{(3)}(t, s_{34}, \tau, \epsilon) &= A^{(2)}(t, s_{34}, \tau, \epsilon) A^{(1)}(t, s_{34}, \tau, \epsilon) - \frac{1}{3}\left[A^{(1)}(t, s_{34}, \tau, \epsilon)\right]^3 \\ &\quad + \frac{4\, G^3(\epsilon)}{G(3\epsilon)} f^{(3)}(\epsilon)\, A^{(1)}(t, s_{34}, \tau, 3\epsilon) + 4\, Const^{(3)} + \mathcal{O}(\epsilon)\,, \end{aligned} \tag{7.10}$$

where the one and two-loop coefficient functions $A^{(1)}(\epsilon)$ and $A^{(2)}(\epsilon)$ are needed to $\mathcal{O}(\epsilon^4)$ and $\mathcal{O}(\epsilon^2)$ respectively.

7.2 Amplitudes in the quasi-multi-Regge kinematics with two pairs, one at each end of the ladder

One can also consider the quasi-multi-Regge kinematics where all gluons are strongly ordered in rapidity, except for two pairs of gluons, one at each end of the ladder,

$$y_3 \simeq y_4 \gg \cdots \gg y_{n-1} \simeq y_n; \qquad |p_{3\perp}| \simeq |p_{4\perp}| \ldots \simeq |p_{n\perp}|\,, \tag{7.11}$$

for which the Mandelstam invariants are given in section D.2 and illustrated in figure 2(b).

The high-energy prescription is

$$m_n(1,2,\ldots,n) = s\left[g^2\,A(p_2,p_3,p_4)\right]\,\frac{1}{t_{n-5}}\,\left(\frac{-s_{n-5}}{\tau}\right)^{\alpha(t_{n-5})}\,\left[g\,V(q_{n-5},q_{n-6},\kappa_{n-6})\right]$$
$$\cdots\times\,\frac{1}{t_2}\,\left(\frac{-s_2}{\tau}\right)^{\alpha(t_2)}\,\left[g\,V(q_2,q_1,\kappa_1)\right]\,\frac{1}{t_1}\,\left(\frac{-s_1}{\tau}\right)^{\alpha(t_1)}\,\left[g^2\,A(p_1,p_n,p_{n-1})\right]. \quad (7.12)$$

where we again suppressed the dependence of the coefficient functions and the Lipatov vertices on the reggeisation scale. In order for m_n to be real, one can take all the invariants s- and t-type to be negative. Then the kinematics imply

$$-s \gg -s_1,-s_2,\ldots,-s_{n-5} \gg -s_{34},-s_{n-1,n},-t_1,-t_2,\ldots,-t_{n-5}\,, \quad (7.13)$$

The limit of multi-Regge kinematics (3.6) is where s_{34} and $s_{n-1,n}$ become as large as any s_i-type invariant.

For $n=6$, eq. (7.12) reduces to two coefficient functions A linked by a t-channel reggeised gluon propagator,

$$m_6(1,2,3,4,5,6) = s\left[g^2\,A(p_2,p_3,p_4,\tau)\right]\,\frac{1}{t}\,\left(\frac{-s_1}{\tau}\right)^{\alpha(t)}\,\left[g^2\,A(p_1,p_n,p_{n-1},\tau)\right]\,, \quad (7.14)$$

with $q = p_1+p_5+p_6 = -(p_2+p_3+p_4)$, $t=q^2$ and $s=s_{12}$. s_1 can be anything between s_{45}, s_{46}, s_{35} and s_{36}, the difference between them being of the order of s_{34} or s_{56}, thus sub-leading with respect to s. The quasi-multi-Regge kinematics (7.3) become

$$-s \gg -s_1 \gg -s_{34},-s_{56},-t\,. \quad (7.15)$$

Expanding eq. (7.14) as in eq. (3.12), at one-, two- and three-loop accuracy, we obtain

$$m_6^{(1)} = \bar\alpha^{(1)}(t)L + \bar A^{(1)}(t,s_{34},\tau) + \bar A^{(1)}(t,s_{56},\tau)\,, \quad (7.16)$$
$$m_6^{(2)} = \frac{1}{2}\left(m_6^{(1)}\right)^2 + \bar\alpha^{(2)}(t)L \quad (7.17)$$
$$+\bar A^{(2)}(t,s_{34},\tau) + \bar A^{(2)}(t,s_{56},\tau) - \frac{1}{2}\left(\bar A^{(1)}(t,s_{34},\tau)\right)^2 - \frac{1}{2}\left(\bar A^{(1)}(t,s_{56},\tau)\right)^2\,,$$
$$m_6^{(3)} = m_6^{(2)}\,m_6^{(1)} - \frac{1}{3}\left(m_6^{(1)}\right)^3 + \bar\alpha^{(3)}(t)L$$
$$+\bar A^{(3)}(t,s_{34},\tau) + \bar A^{(3)}(t,s_{56},\tau)$$
$$-\bar A^{(2)}(t,s_{34},\tau)\bar A^{(1)}(t,s_{34},\tau) - \bar A^{(2)}(t,s_{56}\tau)\bar A^{(1)}(t,s_{56},\tau)$$
$$+\frac{1}{3}\left(\bar A^{(1)}(t,s_{34},\tau)\right)^3 + \frac{1}{3}\left(\bar A^{(1)}(t,s_{56},\tau)\right)^3\,, \quad (7.18)$$

with $L=\ln(-s_1/\tau)$. In the two- and three-loop expansion of the six-point amplitude, (7.17) and (7.18), no new vertices or coefficient functions occur. Thus, using the explicit expressions of $A^{(k)}$ and $\alpha^{(k)}$, $k=1,2,3$, in eq. (7.17) and in eq. (7.18), one can assemble the two- and three-loop six-point amplitude in the quasi-multi-Regge kinematics (7.15). However, even without knowing the explicit expression of $A^{(1)}$ and $A^{(2)}$, it is easy to see by substitution that the iterative structure of eqs. (5.8) and (7.9) ensures that the six-point amplitude (7.17) fulfils the two-loop iterative formula (5.11) for $n = 6$. Similarly, using eq. (7.10), one can easily show that the six-point amplitude (7.18) fulfils the three-loop iterative formula (5.14) for $n = 6$. Thus, also for the quasi-multi-Regge kinematics of eq. (7.15) the quantities $R_6^{(2)}$ and $R_6^{(3)}$ vanish.

Because no new vertices or coefficient functions occur in the two- and three-loop expansion of eq. (7.12) for $n > 6$, we conclude that the two- and three-loop expansions of eq. (7.12) fulfil the two- and three-loop iterative formulas (5.11) and (5.14). Furthermore, it is straightforward to extend the proof of section 6 to the kinematics with a pair of gluons emitted at either side or at each end of the ladder, and hence the BDS ansatz is fulfilled in quasi-multi-Regge kinematics for any n or, in other words, the quantities $R_n^{(l)}$ vanish in the quasi-multi-Regge kinematics (7.11), for any n and for any l.

Continuing the kinematics (7.15) to the physical region where s, s_1, s_{34}, s_{56} are positive and t is negative, the conformal invariants (5.18) become [45]

$$u_1 \simeq 1\,, \quad u_2 \simeq \frac{(|p_{3\perp}|^2 + p_4^+ p_3^-)\, s_{56}}{|q_\perp|^2\,(s_{45}+s_{46})} \simeq \mathcal{O}\left(\frac{t}{s}\right)\,, \quad u_3 = \frac{(|p_{6\perp}|^2 + p_6^+ p_5^-)\, s_{34}}{|q_\perp|^2\,(s_{35}+s_{45})} \simeq \mathcal{O}\left(\frac{t}{s}\right)\,, \tag{7.19}$$

thus, just like for the multi-Regge kinematics (4.26) u_1 is close to 1, while u_2 and u_3 are very small, in fact sub-leading to the desired accuracy.

8. What lies beyond?

From the analysis of sections 5 and 7, it is clear that no difference between the Regge factorisation and the BDS ansatz will be found, unless there is a contribution from coefficient funtions which appear for the first time in n-gluon amplitudes, with $n \geq 6$. To introduce this type of coefficient function means considering even less restrictive multi-Regge kinematics. In this section, we examine the two simplest of such instances: a cluster of two gluons along the ladder, and a cluster of three gluons at one end of the ladder.

8.1 Six-point amplitude in the quasi-multi-Regge kinematics of a pair along the ladder

In the quasi-multi-Regge kinematics of section D.3, where the outgoing gluons are strongly ordered in rapidity, except for the central pair,

$$y_3 \gg y_4 \simeq y_5 \gg y_6; \qquad |p_{3\perp}| \simeq |p_{4\perp}| \simeq |p_{5\perp}| \simeq |p_{6\perp}|\,, \tag{8.1}$$

the high-energy prescription is

$$\begin{aligned} m_6(1,2,3,4,5,6) \;&=\; s\,[g\,C(p_2,p_3,\tau)]\,\frac{1}{t_2}\left(\frac{-s_2}{\tau}\right)^{\alpha(t_2)} \\ &\times \left[g^2\,W(q_2,q_1,p_4,p_5,\tau)\right]\frac{1}{t_1}\left(\frac{-s_1}{\tau}\right)^{\alpha(t_1)}[g\,C(p_1,p_6,\tau)]\,, \end{aligned} \tag{8.2}$$

where $p_4 + p_5 = q_2 - q_1$, and with $t_1 = s_{61}$, $t_2 = s_{23}$, $s_1 = s_{56}$ and $s_2 = s_{34}$ as illustrated in figure 3(a). In order for the amplitude m_6 to be real, eq. (8.2) is taken in the region where all the invariants are negative. Thus, the quasi-multi-Regge kinematics (8.1) become,

$$-s \gg -s_1, -s_2 \gg -s_{45}, -t_1, -t_2 \,. \tag{8.3}$$

In eq. (8.2) a new coefficient function occurs: the vertex for the emission of two gluons along the ladder, $W(q_2, q_1, p_4, p_5, \tau)$, which we shall call the two-gluon Lipatov vertex. Although eq. (8.2) can be defined for a generic helicity configuration, the MHV amplitude requires the two-gluon Lipatov vertex to have two gluons of equal helicity. W can be expanded in the rescaled coupling,

$$\begin{aligned} W(q_2, q_1, p_4, p_5, \tau) &= W^{(0)}(q_2, q_1, p_4, p_5) \\ &\times \left(1 + \bar{g}^2 \bar{W}^{(1)}(t_1, t_2, s_{45}, \tau) + \bar{g}^4 \bar{W}^{(2)}(t_1, t_2, s_{45}, \tau) + \mathcal{O}(\bar{g}^6)\right) \end{aligned} \tag{8.4}$$

The tree approximation, $W^{(0)}(q_2, q_1, p_4, p_5)$, was computed in ref. [47, 48]. The one-loop coefficient, $W^{(1)}(t_1, t_2, s_{45}, \tau)$ is known for the equal-helicity configuration [15]. Expanding eq. (8.2) to one-, two-, and three-loop accuracy, we obtain

$$\begin{aligned} m_6^{(1)} &= \bar{\alpha}^{(1)}(t_1)L_1 + \bar{\alpha}^{(1)}(t_2)L_2 + \bar{C}^{(1)}(t_1, \tau) + \bar{C}^{(1)}(t_2, \tau) + \bar{W}^{(1)}(t_1, t_2, s_{45}, \tau) \\ m_6^{(2)} &= \frac{1}{2}\left(m_6^{(1)}\right)^2 + \bar{\alpha}^{(2)}(t_1)L_1 + \bar{\alpha}^{(2)}(t_2)L_2 \\ &\quad + \bar{C}^{(2)}(t_1, \tau) + \bar{C}^{(2)}(t_2, \tau) + \bar{W}^{(2)}(t_1, t_2, s_{45}, \tau) \\ &\quad - \frac{1}{2}\left(\bar{C}^{(1)}(t_1, \tau)\right)^2 - \frac{1}{2}\left(\bar{C}^{(1)}(t_2, \tau)\right)^2 - \frac{1}{2}\left(\bar{W}^{(1)}(t_1, t_2, s_{45}, \tau)\right)^2 , \end{aligned} \tag{8.5}$$

$$\begin{aligned} m_6^{(3)} &= m_6^{(2)}\, m_6^{(1)} - \frac{1}{3}\left(m_6^{(1)}\right)^3 + \bar{\alpha}^{(3)}(t)L_1 + \bar{\alpha}^{(3)}(t)L_2 \\ &\quad + \bar{C}^{(3)}(t_1, \tau) + \bar{C}^{(3)}(t_2, \tau) + \bar{W}^{(3)}(t_1, t_2, s_{45}, \tau) \\ &\quad - \bar{C}^{(2)}(t_1, \tau)\bar{C}^{(1)}(t_1, \tau) - \bar{C}^{(2)}(t_2, \tau)\bar{C}^{(1)}(t_2, \tau) - \bar{W}^{(2)}(t_1, t_2, s_{45}\tau)\bar{W}^{(1)}(t_1, t_2, s_{45}, \tau) \\ &\quad + \frac{1}{3}\left(\bar{C}^{(1)}(t_1, \tau)\right)^3 + \frac{1}{3}\left(\bar{C}^{(1)}(t_2, \tau)\right)^3 + \frac{1}{3}\left(\bar{W}^{(1)}(t_1, t_2, s_{45}, \tau)\right)^3 , \end{aligned} \tag{8.6}$$

with $L_i = \ln(-s_i/\tau)$ and $i = 1, 2$, and where $m_6^{(1)}$ must be known to $\mathcal{O}(\epsilon^2)$ in eq. (8.5) and $m_6^{(1)}$ and $m_6^{(2)}$ to $\mathcal{O}(\epsilon^4)$ and $\mathcal{O}(\epsilon^2)$ respectively in eq. (8.6). Because for $n = 6$ we expect to find a remainder function $R_6^{(2)}$, combining the iterative formula (1.2) with the two-loop expansion (8.5), we obtain an iterative formula for the vertex $W^{(2)}$,

$$\begin{aligned} W^{(2)}(t_1, t_2, s_{45}, \tau, \epsilon) &= \frac{1}{2}\left[W^{(1)}(t_1, t_2, s_{45}, \tau, \epsilon)\right]^2 \\ &\quad + \frac{2\,G^2(\epsilon)}{G(2\epsilon)} f^{(2)}(\epsilon)\, W^{(1)}(t_1, t_2, s_{45}, \tau, 2\epsilon) + R_6^{(2)}(u_1^W, u_2^W, u_3^W) + \mathcal{O}(\epsilon) , \end{aligned} \tag{8.7}$$

where the one-loop coefficient, $W^{(1)}(\epsilon)$, is needed to $\mathcal{O}(\epsilon^2)$. Thus, a remainder function $R_6^{(2)}$ for the multi-Regge kinematics (8.3) may occur in the two-loop iteration of the two-gluon Lipatov vertex.

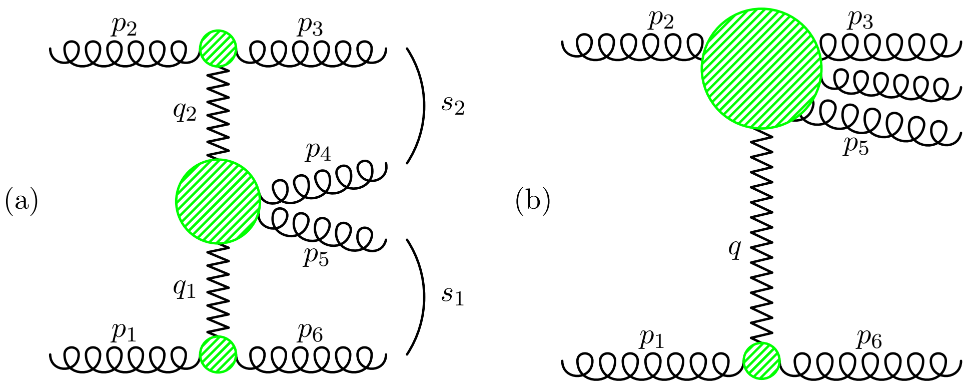

Figure 3: Six-point amplitude in the quasi-multi-Regge kinematics of (a) a pair along the ladder and (b) three-of-a-kind.

Using the Mandelstam invariants of section D.3, the conformal invariants (5.18) become

$$\begin{aligned} u_1 \to u_1^W &= \frac{s_{45}}{(p_4^+ + p_5^+)(p_4^- + p_5^-)} \simeq \mathcal{O}(1)\,, \\ u_2 \to u_2^W &= \frac{|p_{3\perp}|^2 p_5^+ p_6^-}{(|p_{3\perp} + p_{4\perp}|^2 + p_5^+ p_4^-)(p_4^+ + p_5^+)p_6^-} \simeq \mathcal{O}(1)\,, \\ u_3 \to u_3^W &= \frac{|p_{6\perp}|^2 p_3^+ p_4^-}{p_3^+(p_4^- + p_5^-)(|p_{3\perp} + p_{4\perp}|^2 + p_5^+ p_4^-)} \simeq \mathcal{O}(1)\,, \end{aligned} \tag{8.8}$$

i.e. all the invariants yield a non-vanishing contribution, which is in general different from unity.

8.2 Six-point amplitude in the quasi-multi-Regge kinematics of three-of-a-kind

In the quasi-multi-Regge kinematics of section D.4, where the outgoing gluons are emitted three in a cluster on one end and one on the other end of the ladder,

$$y_3 \simeq y_4 \simeq y_5 \gg y_6; \qquad |p_{3\perp}| \simeq |p_{4\perp}| \simeq |p_{5\perp}| \simeq |p_\perp|\,, \tag{8.9}$$

the high-energy prescription is

$$m_6(1,2,3,4,5,6) \,=\, s\,[g\,B(p_2,p_3,p_4,p_5,\tau)]\,\frac{1}{t}\left(\frac{-s_1}{\tau}\right)^{\alpha(t)}[g\,C(p_1,p_6,\tau)]\;, \tag{8.10}$$

where $q = p_1 + p_6$, as shown in figure 3(b), $t = q^2$ and $s = s_{12}$. s_1 can be anything between s_{36}, s_{46}, and s_{56}, the difference between them being of the order of s_{345}, thus sub-leading with respect to s. In order for the amplitude m_6 to be real, eq. (8.10) is taken in the region where all the invariants are negative. Thus, the quasi-multi-Regge kinematics (8.9) become,

$$-s \gg -s_1 \gg -s_{34}, -s_{45}, -s_{35}, -t\,. \tag{8.11}$$

In eq. (8.10) a new coefficient function occurs for the emission of three gluons at one end of the ladder occurs, $B(p_3, p_4, p_5, \tau)$. B can be expanded in the rescaled coupling,

$$B(p_3, p_4, p_5, \tau) = B^{(0)}(p_3, p_4, p_5) \times \left(1 + \bar{g}^2 \bar{B}^{(1)}(t, s_{34}, s_{45}, s_{35}, \tau) + \bar{g}^4 \bar{B}^{(2)}(t, s_{34}, s_{45}, s_{35}, \tau) + \mathcal{O}(\bar{g}^6)\right) . \tag{8.12}$$

The tree approximation, $B^{(0)}(p_3, p_4, p_5)$, was computed in ref. [49]. Expanding eq. (8.10) to one-, two- and three-loop accuracy, we obtain

$$\begin{aligned}
m_6^{(1)} &= \bar{\alpha}^{(1)}(t)L + \bar{B}^{(1)}(t, s_{34}, s_{45}, s_{35}, \tau) + \bar{C}^{(1)}(t, \tau) , \\
m_6^{(2)} &= \frac{1}{2}\left(m_6^{(1)}\right)^2 + \bar{\alpha}^{(2)}(t)L + \bar{B}^{(2)}(t, s_{34}, s_{45}, s_{35}, \tau) + \bar{C}^{(2)}(t, \tau) \qquad (8.13) \\
&\quad -\frac{1}{2}\left(\bar{B}^{(1)}(t, s_{34}, s_{45}, s_{35}, \tau)\right)^2 - \frac{1}{2}\left(\bar{C}^{(1)}(t, \tau)\right)^2 , \\
m_6^{(3)} &= m_6^{(2)} m_6^{(1)} - \frac{1}{3}\left(m_6^{(1)}\right)^3 + \bar{\alpha}^{(3)}(t)L + \bar{B}^{(3)}(t, s_{34}, s_{45}, s_{35}, \tau) + \bar{C}^{(3)}(t, \tau) \qquad (8.14) \\
&\quad -\bar{B}^{(2)}(t, s_{34}, s_{45}, s_{35}, \tau)\bar{B}^{(1)}(t, s_{34}, s_{45}, s_{35}, \tau) - \bar{C}^{(2)}(t, \tau)\bar{C}^{(1)}(t, \tau) \\
&\quad +\frac{1}{3}\left(\bar{B}^{(1)}(t, s_{34}, s_{45}, s_{35}, \tau)\right)^3 + \frac{1}{3}\left(\bar{C}^{(1)}(t, \tau)\right)^3 ,
\end{aligned}$$

with $L = \ln(-s_1/\tau)$, and where $m_6^{(1)}$ must be known to $\mathcal{O}(\epsilon^2)$ in eq. (8.13) and $m_6^{(1)}$ and $m_6^{(2)}$ to $\mathcal{O}(\epsilon^4)$ and to $\mathcal{O}(\epsilon^2)$ respectively in eq. (8.13). Because for $n = 6$ we expect to find a remainder function $R_6^{(2)}$, combining the iterative formula (1.2) with the two-loop expansion (8.13), we obtain an iterative formula for the vertex $B^{(2)}$,

$$\begin{aligned}
B^{(2)}(t, s_{34}, s_{45}, s_{35}, \tau, \epsilon) &= \frac{1}{2}\left[B^{(1)}(t, s_{34}, s_{45}, s_{35}, \tau, \epsilon)\right]^2 \\
&\quad + \frac{2\,G^2(\epsilon)}{G(2\epsilon)} f^{(2)}(\epsilon)\, B^{(1)}(t, s_{34}, s_{45}, s_{35}, \tau, 2\epsilon) + 2\,Const^{(2)} \\
&\quad + R_6^{(2)}(u_1^B, u_2^B, u_3^B) + \mathcal{O}(\epsilon) ,
\end{aligned} \tag{8.15}$$

where the one-loop coefficient, $B^{(1)}(\epsilon)$, is needed to $\mathcal{O}(\epsilon^2)$. Thus, a remainder function $R_6^{(2)}$ for the multi-Regge kinematics (8.11) may occur in the two-loop iteration for the coefficient function for the emission of three gluons on one end of the ladder.

In the limit $y_3 \gg y_4 \simeq y_5$, the kinematics (8.9) reduce to eq. (8.1) and the prescription (8.10) reduces to eq. (8.2). Then the coefficient function B factors out into the two-gluon Lipatov vertex W and the coefficient function for the emission of a gluon, linked by a reggeised propagator [49]. Accordingly, the remainder function $R_6^{(2)}(u_1^B, u_2^B, u_3^B)$ in eq. (8.15) reduces to $R_6^{(2)}(u_1^W, u_2^W, u_3^W)$ in eq. (8.7).

Using the Mandelstam invariants of section D.4, the conformal invariants (5.18) become [45]

$$
\begin{aligned}
u_1 \to u_1^B &= \frac{s\, s_{45}}{s_{345}(p_4^+ + p_5^+)p_6^-} \simeq \mathcal{O}(1)\,, \\
u_2 \to u_2^B &= \frac{(|p_{3\perp}|^2 + (p_4^+ + p_5^+)p_3^-)p_5^+ p_6^-}{(p_4^+ + p_5^+)p_6^-(|p_{3\perp} + p_{4\perp}|^2 + (p_3^- + p_4^-)p_5^+)} \simeq \mathcal{O}(1)\,, \\
u_3 \to u_3^B &= \frac{|p_{6\perp}|^2 s_{34}}{s_{345}(|p_{3\perp} + p_{4\perp}|^2 + (p_3^- + p_4^-)p_5^+)} \simeq \mathcal{O}(1)\,,
\end{aligned} \tag{8.16}
$$

i.e. all the invariants are of similar size.

9. Conclusions

In this work we investigated the high-energy limit of a colour-stripped MHV amplitude, which is based on the Regge factorisation of the amplitude into a ladder of coefficient functions and vertices linked by reggeised propagators [26]. We showed explicitly that two- and three-loop n-gluon amplitudes in multi-Regge kinematics are fully consistent with the Bern-Dixon-Smirnov ansatz, and in section 6 we proved that this result holds true at any loop accuracy. In particular, this implies that the breakdown of the iterative structure of the two-loop amplitudes, occurring in the two-loop six-point amplitude, cannot be resolved by multi-Regge kinematics, i.e. the remainder function $R_6^{(2)}$ is sub-leading in the multi-Regge kinematics.

In section 7 we showed that similar conclusions can be drawn for less restrictive multi-Regge kinematics, namely the kinematics where all the outgoing gluons are strongly ordered in rapidity, but for a pair of gluons either at one end or at both ends of the ladder. By giving explicit examples for the two- and three-loop six-point amplitude, we argued that in this case as well the Regge factorisation of the amplitude is consistent with the iterative structure implied by the BDS ansatz. The structure of the high energy prescription ensures that this result is valid for an arbitrary number of loops.

Finally, in order to find kinematics which might shed light on the violation of the BDS ansatz for the two-loop six-point amplitude, in section 8 we considered kinematics which occur only for n-gluon amplitudes with $n \geq 6$, and thus for which we could not invoke the BDS iterative structure. We showed that the iterative structures for the new two-loop functions that appear in these kinematics might have a dependence on the remainder function $R_6^{(2)}(u_1, u_2, u_3)$, where u_1, u_2, u_3 are the conformal invariants, and therefore we argued that these kinematical limits could provide some information on this quantity. This suggestion is supported by the observation that, while in the multi-Regge kinematics of section 5 and in the quasi-multi-Regge kinematics of section 7 the three conformal cross ratios (5.18) all took limiting values, in the more general quasi-multi-Regge kinematics of section 8 they are allowed to vary over a range defined by the kinematic invariants.

Acknowledgments

We thank Lance Dixon, Vladimir Smirnov and Gabriele Travaglini for useful discussions. CD thanks the IPPP Durham and the LNF Frascati for the warm hospitality at various stages of this work. CD is a research fellow of the *Fonds National de la Recherche Scientifique*, Belgium. This work was partly supported by MIUR under contract $2006020509_0 04$, and by the EC Marie-Curie Research Training Network "Tools and Precision Calculations for Physics Discoveries at Colliders" under contract MRTN-CT-2006-035505. EWNG gratefully acknowledges the support of the Wolfson Foundation and the Royal Society.

A. Multi-parton kinematics

We consider the production of $n-2$ gluons of momentum p_i, with $i = 3, \ldots, n$ in the scattering between two partons of momenta p_1 and p_2.[5]

Using light-cone coordinates $p^{\pm} = p_0 \pm p_z$, and complex transverse coordinates $p_\perp = p^x + ip^y$, with scalar product $2p \cdot q = p^+ q^- + p^- q^+ - p_\perp q_\perp^* - p_\perp^* q_\perp$, the 4-momenta are,

$$\begin{aligned} p_2 &= \left(p_2^+/2, 0, 0, p_2^+/2\right) = \left(p_2^+, 0; 0, 0\right) \ , \\ p_1 &= \left(p_1^-/2, 0, 0, -p_1^-/2\right) = \left(0, p_1^-; 0, 0\right) \ , \\ p_i &= \left((p_i^+ + p_i^-)/2, \mathrm{Re}[p_{i\perp}], \mathrm{Im}[p_{i\perp}], (p_i^+ - p_i^-)/2\right) \\ &= \left(|p_{i\perp}| e^{y_i}, |p_{i\perp}| e^{-y_i}; |p_{i\perp}| \cos\phi_i, |p_{i\perp}| \sin\phi_i\right) \ , \end{aligned} \tag{A.1}$$

where y is the rapidity. The first notation above is the standard representation $p^\mu = (p^0, p^x, p^y, p^z)$, while in the second we have the + and - components on the left of the semicolon, and on the right the transverse components. In the following, if not differently stated, p_i and p_j are always understood to lie in the range $3 \le i, j \le n$. The mass-shell condition is $|p_{i\perp}|^2 = p_i^+ p_i^-$. From the momentum conservation,

$$\begin{aligned} 0 &= \sum_{i=3}^{n} p_{i\perp} \, , \\ p_2^+ &= -\sum_{i=3}^{n} p_i^+ \, , \\ p_1^- &= -\sum_{i=3}^{n} p_i^- \, , \end{aligned} \tag{A.2}$$

the Mandelstam invariants may be written as,

$$s_{ij} = 2p_i \cdot p_j = p_i^+ p_j^- + p_i^- p_j^+ - p_{i\perp} p_{j\perp}^* - p_{i\perp}^* p_{j\perp} \, , \tag{A.3}$$

so that

$$\begin{aligned} s &= 2p_1 \cdot p_2 = \sum_{i,j=3}^{n} p_i^+ p_j^- \, , \\ s_{2i} &= 2p_2 \cdot p_i = -\sum_{j=3}^{n} p_i^- p_j^+ \, , \\ s_{1i} &= 2p_1 \cdot p_i = -\sum_{j=3}^{n} p_i^+ p_j^- \, . \end{aligned} \tag{A.4}$$

[5]By convention we consider the scattering in the unphysical region where all momenta are taken as outgoing, and then we analitically continue to the physical region where $p_1^0 < 0$ and $p_2^0 < 0$.

Using the spinor representation of ref. [49],

$$\psi_+(p_i) = \begin{pmatrix} \sqrt{p_i^+} \\ \sqrt{p_i^-}\, e^{i\phi_i} \\ 0 \\ 0 \end{pmatrix}, \quad \psi_-(p_i) = \begin{pmatrix} 0 \\ 0 \\ \sqrt{p_i^-}\, e^{-i\phi_i} \\ -\sqrt{p_i^+} \end{pmatrix},$$

$$\psi_+(p_2) = i \begin{pmatrix} \sqrt{-p_2^+} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \psi_-(p_2) = i \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\sqrt{-p_2^+} \end{pmatrix}, \tag{A.5}$$

$$\psi_+(p_1) = -i \begin{pmatrix} 0 \\ \sqrt{-p_1^-} \\ 0 \\ 0 \end{pmatrix}, \psi_-(p_1) = -i \begin{pmatrix} 0 \\ 0 \\ \sqrt{-p_1^-} \\ 0 \end{pmatrix}.$$

for the momenta (A.1),[6] the spinor products are

$$\begin{aligned} \langle 21 \rangle &= -\sqrt{s}\,, \\ \langle 2i \rangle &= -i\sqrt{\frac{-p_2^+}{p_i^+}}\, p_{i\perp}\,, \\ \langle i1 \rangle &= i\sqrt{-p_1^- p_i^+}\,, \\ \langle ij \rangle &= p_{i\perp}\sqrt{\frac{p_j^+}{p_i^+}} - p_{j\perp}\sqrt{\frac{p_i^+}{p_j^+}}\,, \end{aligned} \tag{A.6}$$

where we have used the mass-shell condition $|p_{i\perp}|^2 = p_i^+ p_i^-$. The spinor products fulfill the usual identities,

$$\begin{aligned} \langle ij \rangle &= -\langle ji \rangle \\ [ij] &= -\,[ji] \\ \langle ij \rangle^* &= \mathrm{sign}(p_i^0 p_j^0)\,[ji] \\ (\langle i+|\gamma^\mu|j+\rangle)^* &= \mathrm{sign}(p_i^0 p_j^0)\langle j+|\gamma^\mu|i+\rangle \\ \langle ij \rangle\,[ji] &= 2p_i \cdot p_j = \hat{s}_{ij} \\ \langle i+|\not{k}|j+\rangle &= [ik]\,\langle kj \rangle \\ \langle i-|\not{k}|j-\rangle &= \langle ik \rangle\,[kj] \\ \langle ij \rangle\langle kl \rangle &= \langle ik \rangle\langle jl \rangle + \langle il \rangle\langle kj \rangle \end{aligned} \tag{A.7}$$

[6] The spinors of the incoming partons must be continued to negative energy after the complex conjugation, e.g. $\overline{\psi_+(p_2)} = i\left(\sqrt{-p_2^+}, 0, 0, 0\right)$.

and if $\sum_{i=1}^{n} p_i = 0$ then

$$\sum_{i=1}^{n} [ji] \langle ik \rangle = 0 \,. \tag{A.8}$$

B. Multi-Regge kinematics

In the multi-Regge kinematics, we require that the gluons are strongly ordered in rapidity and have comparable transverse momentum (2.1). This is equivalent to require a strong ordering of the light-cone coordinates,

$$p_3^+ \gg p_4^+ \cdots \gg p_n^+ ; \qquad p_3^- \ll p_4^- \cdots \ll p_n^- \,. \tag{B.1}$$

In the high-energy limit, momentum conservation (A.2) then becomes

$$\begin{aligned} 0 &= \sum_{i=3}^{n} p_{i\perp} \,, \\ p_2^+ &= -p_3^+ \,, \\ p_1^- &= -p_n^- \,, \end{aligned} \tag{B.2}$$

where the = sign is understood to mean "equals up to corrections of next-to-leading accuracy". The Mandelstam invariants (A.4) are reduced to,

$$\begin{aligned} s &= 2p_1 \cdot p_2 = p_3^+ p_n^- , \\ s_{2i} &= 2p_2 \cdot p_i = -p_3^+ p_i^- , \\ s_{1i} &= 2p_1 \cdot p_i = -p_i^+ p_n^- , \\ s_{ij} &= 2p_i \cdot p_j = p_i^+ p_j^- \qquad i < j \,. \end{aligned} \tag{B.3}$$

The product of two successive invariants of type s_{ij} fixes the mass shell. For example,

$$s_{k-1,k} s_{k,k+1} = p_{k-1}^+ p_k^- p_k^+ p_{k+1}^- = |p_{k\perp}|^2 p_{k-1}^+ p_{k+1}^- = |p_{k\perp}|^2 s_{k-1,k+1} = |p_{k\perp}|^2 s_{k-1,k,k+1} \,.$$

Thus,

$$|p_{k\perp}|^2 = \frac{s_{k-1,k} s_{k,k+1}}{s_{k-1,k,k+1}} \,. \tag{B.4}$$

The spinor products (A.6) are,

$$\begin{aligned} \langle 21 \rangle &= -\sqrt{p_3^+ p_n^-} \,, \\ \langle 2i \rangle &= -i \sqrt{\frac{p_3^+}{p_i^+}}\, p_{i\perp} \,, \\ \langle i1 \rangle &= i \sqrt{p_i^+ p_n^-} \,, \\ \langle ij \rangle &= -\sqrt{\frac{p_i^+}{p_j^+}}\, p_{j\perp} \qquad \text{for}\, y_i > y_j \,. \end{aligned} \tag{B.5}$$

B.1 6-point amplitude in multi-Regge kinematics

For $n = 6$, the momenta of the gluons exchanged in the t channel are $q_1 = p_1 + p_6$, $q_2 = q_1 + p_5 = q_3 - p_4$, $q_3 = -p_2 - p_3$. The cyclic Mandelstam invariants are

$$\begin{aligned} s &= p_3^+ p_6^- , \\ s_{23} &= -p_3^+ p_3^- = -|p_{3\perp}|^2 = -|q_{3\perp}|^2 , \\ s_{34} &= p_3^+ p_4^- , \\ s_{45} &= p_4^+ p_5^- , \\ s_{56} &= p_5^+ p_6^- , \\ s_{61} &= -p_6^+ p_6^- = -|p_{6\perp}|^2 = -|q_{1\perp}|^2 . \end{aligned} \tag{B.6}$$

Then we see that

$$s_{345}\, s_{456} = s\, s_{45} . \tag{B.7}$$

The mass-shell conditions for the gluons along the ladder imply that

$$\begin{aligned} s_{34}\, s_{45} &= s_{345}\, |p_{4\perp}|^2 = s_{345}\, |q_{3\perp} - q_{2\perp}|^2 , \\ s_{45}\, s_{56} &= s_{456}\, |p_{5\perp}|^2 = s_{456}\, |q_{2\perp} - q_{1\perp}|^2 . \end{aligned} \tag{B.8}$$

The mass-shell conditions and eq. (B.7) imply that

$$s_{34}\, s_{45}\, s_{56} = s\, |p_{4\perp}|^2\, |p_{5\perp}|^2 . \tag{B.9}$$

In addition, one can see that

$$s_{23} + s_{34} + s_{24} = -|p_{3\perp} + p_{4\perp}|^2 = -|q_{2\perp}|^2 . \tag{B.10}$$

The momentum that flows out along the ladder is $p_4 + p_5 = q_3 - q_1$, with

$$|p_{4\perp} + p_{5\perp}|^2 = s_{45} \left(1 - \frac{s\, s_{45}}{s_{345}\, s_{456}} \right) . \tag{B.11}$$

C. High-energy limit vs. ϵ-expansion

In ref. [15], the six-point one-loop amplitude was analyzed, and it was pointed out that the result for the one-loop n-point amplitude obtained in ref. [50] contains dilogarithms for $n \geq 6$, whereas in multi-Regge kinematics, the n-point amplitude is built up completely from coefficient functions and Lipatov vertices. Those building blocks are themselves fully determined by the five-point amplitude, where no dilogarithms appear. In particular, in ref. [15] the function $\mathrm{Li}_2\,(1 - \Phi)$, with

$$\Phi = \frac{(-s)(-s_2)}{(-s_{345})(-s_{456})} , \tag{C.1}$$

was considered. It is easy to see that in the euclidean region in multi-Regge kinematics Φ becomes

$$\Phi \equiv 1 \Rightarrow \mathrm{Li}_2\,(1 - \Phi) \equiv 0 , \tag{C.2}$$

i.e. the dilogarithm disappears in the euclidean region in multi-Regge kinematics. The same holds true in the physical region where all s-type invariants are positive. The authors claim however that there is a problem when going to the region where $s, s_2 > 0$, all other invariants staying negative. Although this region is unphysical, it implies that there is a problem with factorization because starting from the euclidean region in multi-Regge kinematics one could not reach all the other regions by simple analytic continuation. Their argument goes as follows: The analytic continuation to this region is performed in the usual way by continuing s and s_2 along a half circle in the upper half plane,

$$(-s_2) \to e^{-i\pi} s_2, \qquad (-s) \to e^{-i\pi} s. \tag{C.3}$$

As all other invariants in this region stay negative, this implies

$$\Phi \to e^{-2i\pi}\Phi, \tag{C.4}$$

and using the analytic continuation properties of the dilogarithm, it is easy to see that in this region the dilogarithm develops an imaginary part,

$$\mathrm{Li}_2\left(1-\Phi\right) \to \mathrm{Li}_2\left(1-e^{-2i\pi}\Phi\right) \equiv 2i\pi\, \ln\left(1-\Phi\right), \tag{C.5}$$

i.e. , the dilogarithm would not completely vanish in this region but leave a trace in form of this imaginary part, which would be absent when we first computed the multi-Regge limit in the euclidean region, and then continued this function to the region we consider. On the other hand, in the multi-Regge limit this imaginary part becomes $2i\pi\, \ln 0$, i.e. , an expression which is ill-defined.

In the following we are going to show that this ill-defined imaginary part arises due to a non commutation between the high-energy limit we consider and the expansion in ϵ, and more specifically the expansion of the 2 mass easy (2me) box. In the rest of this section we will analyse the high-energy limit of the 2me box, and we will show explicitly how this imaginary part arises due to the non commutation of the multi-Regge limit with the ϵ expansion.

C.1 The 2me box expanded in ϵ

It is easy to see that the term proportional to $\mathrm{Li}_2\left(1-\Phi\right)$ comes from the contribution of the ϵ expansion of the 2me box. In the euclidean region, the 2me box can be written as [50]

$$\mu^{2\epsilon}\, I_4^{2me}(s,t,P^2,Q^2) = \frac{2c_\Gamma\, \mu^{2\epsilon}}{P^2Q^2 - st}\, F^{2me}(s,t,P^2,Q^2), \tag{C.6}$$

where s, t denote the Mandelstam invariants, and P^2, Q^2 denote the masses of the two massive legs, and

$$\begin{aligned}
\mu^{2\epsilon}\,F^{2me}(s,t,P^2,Q^2) =& -\frac{1}{\epsilon^2}\left[\left(\frac{-s}{\mu^2}\right)^{-\epsilon}+\left(\frac{-t}{\mu^2}\right)^{-\epsilon}-\left(\frac{-P^2}{\mu^2}\right)^{-\epsilon}-\left(\frac{-Q^2}{\mu^2}\right)^{-\epsilon}\right]\\
&+\mathrm{Li}_2\left(1-\frac{P^2}{s}\right)+\mathrm{Li}_2\left(1-\frac{Q^2}{s}\right)+\mathrm{Li}_2\left(1-\frac{P^2}{t}\right)+\mathrm{Li}_2\left(1-\frac{Q^2}{t}\right)\\
&-\mathrm{Li}_2\left(1-\frac{P^2Q^2}{st}\right)+\frac{1}{2}\ln^2\left(\frac{s}{t}\right)+\mathcal{O}(\epsilon)\\
=&\frac{1}{\epsilon}\ln\left(\frac{st}{P^2Q^2}\right)+\frac{1}{2}\left[\ln^2\left(\frac{-P^2}{\mu^2}\right)+\ln^2\left(\frac{-Q^2}{\mu^2}\right)-\ln^2\left(\frac{-s}{\mu^2}\right)-\ln^2\left(\frac{-t}{\mu^2}\right)\right]\\
&+\mathrm{Li}_2\left(1-\frac{P^2}{s}\right)+\mathrm{Li}_2\left(1-\frac{Q^2}{s}\right)+\mathrm{Li}_2\left(1-\frac{P^2}{t}\right)+\mathrm{Li}_2\left(1-\frac{Q^2}{t}\right)\\
&-\mathrm{Li}_2\left(1-\frac{P^2Q^2}{st}\right)+\frac{1}{2}\ln^2\left(\frac{s}{t}\right)+\mathcal{O}(\epsilon)
\end{aligned} \tag{C.7}$$

The masses Q^2 and P^2 and the Mandelstam invariants s and t can be related to the two and three-particle invariants in a straightforward way,

$$Q^2 = s, \quad P^2 = s_2, \quad s = s_{345}, \quad t = s_{456}. \tag{C.8}$$

The multi-Regge limit we are interested in corresponds to the limit defined by the rescaling

$$(-Q^2) \sim \frac{1}{\lambda^2}, \qquad (-s),(-t) \sim \frac{1}{\lambda}, \qquad (-P^2) \sim 1, \tag{C.9}$$

where $\lambda \to 0$. Furthermore, eq. (B.7) imposes the constraint

$$Q^2P^2 \equiv st. \tag{C.10}$$

We can expand the dilogarithms that appear in eq. (C.7) in this limit,

$$\begin{aligned}
\mathrm{Li}_2\left(1-\frac{P^2}{s}\right) =&\zeta_2+\mathcal{O}\left(\frac{P^2}{s}\right),\\
\mathrm{Li}_2\left(1-\frac{P^2}{t}\right) =&\zeta_2+\mathcal{O}\left(\frac{P^2}{t}\right),\\
\mathrm{Li}_2\left(1-\frac{Q^2}{s}\right) =&-\zeta_2-\frac{1}{2}\ln^2\left(\frac{Q^2}{s}\right)+\mathcal{O}\left(\frac{s}{Q^2}\right),\\
\mathrm{Li}_2\left(1-\frac{Q^2}{t}\right) =&-\zeta_2-\frac{1}{2}\ln^2\left(\frac{Q^2}{t}\right)+\mathcal{O}\left(\frac{t}{Q^2}\right),
\end{aligned} \tag{C.11}$$

and it is straightforward to check that we find

$$\mu^{2\epsilon}\,F^{2me}(s,t,P^2,Q^2) \equiv 0. \tag{C.12}$$

Let us turn to the region defined by $s,t < 0$ and $P^2,Q^2 > 0$, which corresponds to the region analysed in ref. [15] for the full six-point amplitude. If we perform the analytic continuation of eq. (C.7) to this region according to the standard prescription

$\ln(-s + i\varepsilon) = \ln |s| - i\,\pi\,\theta(s)$, and then approach the multi-Regge limit, we find a non-vanishing imaginary part,

$$\mu^{2\epsilon}\, F^{2me}(s,t,P^2,Q^2) \equiv \frac{2\pi i}{\epsilon} - 2\pi i \ln\left(\frac{P^2}{\mu^2}\right) - 2\pi i \ln\left(1 - \frac{P^2Q^2}{st}\right) + \mathcal{O}(\epsilon), \tag{C.13}$$

in agreement with the result obtained in ref. [15]. Note that we could not have obtained this imaginary part by analytically continuing the multi-Regge expression of the 2me box in the euclidean region, eq. (C.12), which vanished identically. We will now show that taking the multi-Regge limit before making the expansion in ϵ leads to consistent results in all kinematic regions.

C.2 The 2me box to all orders in ϵ

In this section we repeat the analysis of the multi-Regge limit of the 2me box, but taking the limit on the all orders in ϵ-expression instead of the ϵ-expanded form. The expression of the 2me box in the euclidean region to all orders in ϵ can be found in ref. [51],

$$\begin{aligned}\mu^{2\epsilon}\, F^{2me}(s,t,P^2,Q^2) &= -\frac{1}{\epsilon^2}\Bigg\{ \left(\frac{-s}{\mu^2}\right)^{-\epsilon} \left[1 - {}_2F_1\left(1,\epsilon,1+\epsilon;\frac{1}{as}\right)\right] \\ &+ \left(\frac{-t}{\mu^2}\right)^{-\epsilon} \left[1 - {}_2F_1\left(1,\epsilon,1+\epsilon;\frac{1}{at}\right)\right] - \left(\frac{-P^2}{\mu^2}\right)^{-\epsilon} \left[1 - {}_2F_1\left(1,\epsilon,1+\epsilon;\frac{1}{aP^2}\right)\right] \\ &- \left(\frac{-Q^2}{\mu^2}\right)^{-\epsilon} \left[1 - {}_2F_1\left(1,\epsilon,1+\epsilon;\frac{1}{aQ^2}\right)\right]\Bigg\},\end{aligned} \tag{C.14}$$

where

$$a = \frac{u}{P^2Q^2 - st} \qquad \text{and} \qquad u = P^2 + Q^2 - s - t, \tag{C.15}$$

and hence in the multi-Regge limit $u \equiv Q^2$ and $a \to \infty$. Using the scaling (C.9), it is easy to see that the leading contribution comes from the term proportional to $(-P^2)^{-\epsilon}$, because all other terms are suppressed by at least one additional power of λ. Hence, in the multi-Regge limit we have

$$\mu^{2\epsilon}\, F^{2me}(s,t,P^2,Q^2) \equiv \frac{1}{\epsilon^2}\left(\frac{-P^2}{\mu^2}\right)^{-\epsilon} \left[1 - {}_2F_1\left(1,\epsilon,1+\epsilon;1-\tilde{\Phi}\right)\right], \tag{C.16}$$

where we defined $\tilde{\Phi} \equiv \frac{st}{P^2Q^2}$, and the limit is now reached by $\tilde{\Phi} \to 1$. As the hypergeometric function is continuous close to the origin, we get

$$\lim_{\tilde{\Phi}\to 1} {}_2F_1\left(1,\epsilon,1+\epsilon;1-\tilde{\Phi}\right) = 1, \tag{C.17}$$

and so in the euclidean region the 2me box vanishes, in agreement with the result obtained in the previous section.

Let us turn to the region defined by $s, t < 0$ and $P^2, Q^2 > 0$. When going to this region we have to perform the analytic continuation of $\tilde{\Phi}$ according to $\tilde{\Phi} \to e^{2\pi i}\tilde{\Phi}$. However, in this region we cannot directly use the expression (C.16), but we first need to perform the analytic continuation of the hypergeometric function to the region we consider. The ϵ-expansion of the hypergeometric function in the euclidean region is given by

$$ {}_2F_1(1,\epsilon,1+\epsilon;1-\tilde{\Phi}) = 1 - \sum_{n=1}^{\infty} (-\epsilon)^n \, \mathrm{Li}_n(1-\tilde{\Phi}), \tag{C.18} $$

and hence

$$ {}_2F_1(1,\epsilon,1+\epsilon;1-e^{2\pi i}\tilde{\Phi}) = 1 - \sum_{n=1}^{\infty} (-\epsilon)^n \, \mathrm{Li}_n(1-e^{2\pi i}\tilde{\Phi}). \tag{C.19} $$

Since we know the analytic properties of the polylogarithms,

$$ \mathrm{Im}\,\mathrm{Li}_n(1-e^{2\pi i}\tilde{\Phi}) = -2\pi \frac{\ln^{n-1}(1-\tilde{\Phi})}{(n-1)!}, \tag{C.20} $$

we deduce

$$ \begin{aligned} \mathrm{Im}\,{}_2F_1(1,\epsilon,1+\epsilon;1-e^{2\pi i}\tilde{\Phi}) =&\ 2\pi \sum_{n=1}^{\infty} (-\epsilon)^n \, \frac{\ln^{n-1}(1-\tilde{\Phi})}{(n-1)!} \\ =& -2\pi\epsilon \sum_{n=0}^{\infty} (-\epsilon)^n \, \frac{\ln^{n}(1-\tilde{\Phi})}{n!} \ = \ -2\pi\epsilon\,(1-\tilde{\Phi})^{-\epsilon}, \end{aligned} \tag{C.21} $$

and hence

$$ {}_2F_1(1,\epsilon,1+\epsilon;1-e^{2\pi i}\tilde{\Phi}) = {}_2F_1(1,\epsilon,1+\epsilon;1-\tilde{\Phi}) - 2\pi\, i\,\epsilon\,(1-\tilde{\Phi})^{-\epsilon}. \tag{C.22} $$

From this we immediately see that in the limit $\tilde{\Phi} \to 1$,

$$ {}_2F_1(1,\epsilon,1+\epsilon;1-e^{2\pi i}\tilde{\Phi}) \to 1, \tag{C.23} $$

and therefore the 2me box vanishes prior to making the expansion in ϵ. This is traced back to the imaginary part which behaves as,

$$ \lim_{\epsilon\to 0} \lim_{\tilde{\Phi}\to 1} \frac{2\pi\, i}{\epsilon} \,(1-\tilde{\Phi})^{-\epsilon} \equiv 0. \tag{C.24} $$

By taking the $\tilde{\Phi} \to 1$ limit first, we find complete agreement with what we would have obtained by analytically continuing the limit from the euclidean region to the new region. On the other hand, if we expand in ϵ prior to taking the $\tilde{\Phi} \to 1$ limit, one produces an imaginary part in eq. (C.22),

$$ \lim_{\tilde{\Phi}\to 1} \lim_{\epsilon\to 0} \frac{2\pi\, i}{\epsilon} \,(1-\tilde{\Phi})^{-\epsilon} = \lim_{\tilde{\Phi}\to 1} \frac{2\pi i}{\epsilon} - 2\pi i \ln(1-\tilde{\Phi}) + \mathcal{O}(\epsilon). \tag{C.25} $$

This is precisely the same imaginary part that we obtained in eq. (C.5). It is clear that the two limits do not commute. The consistent approach is to take the multi-Regge limit $\tilde{\Phi} \to 1$ first.

JHEP12(2008)097

D. Quasi multi-Regge kinematics

D.1 Quasi-multi-Regge kinematics of a pair at either end of the ladder

In the quasi-multi-Regge kinematics of eq. (7.1), we require that the gluons are strongly ordered in rapidity, except for a pair at either end of the ladder. In light-cone coordinates, it is

$$p_3^+ \simeq p_4^+ \gg p_5^+ \cdots \gg p_n^+ ; \qquad p_3^- \simeq p_4^- \ll p_5^- \cdots \ll p_n^- . \tag{D.1}$$

Momentum conservation (A.2) then becomes

$$\begin{aligned} 0 &= \sum_{i=3}^{n} p_{i\perp} , \\ p_2^+ &= -(p_3^+ + p_4^+) , \\ p_1^- &= -p_n^- . \end{aligned} \tag{D.2}$$

The cyclic Mandelstam invariants are

$$\begin{aligned} s &= (p_3^+ + p_4^+) p_n^- , \\ s_{23} &= -(p_3^+ + p_4^+) p_3^- = -|p_{3\perp}|^2 - p_4^+ p_3^- , \\ s_{45} &= p_4^+ p_5^- , \\ &\vdots \\ s_{n-1,n} &= p_{n-1}^+ p_n^- , \\ s_{n1} &= -p_n^+ p_n^- = -|p_{n\perp}|^2 , \end{aligned} \tag{D.3}$$

where we did not indicate s_{34} because it is written as in eq. (A.3), since no approximation is taken on it.

D.2 Quasi-multi-Regge kinematics of two pairs, one at each end of the ladder

In the quasi-multi-Regge kinematics of eq. (7.11), we require that the gluons are strongly ordered in rapidity, except for two pairs, one at each end of the ladder. In light-cone coordinates, it is

$$\begin{aligned} p_3^+ &\simeq p_4^+ \gg p_5^+ \cdots \gg p_{n-2} + \gg p_{n-1}^+ \simeq p_n^+ , \\ p_3^- &\simeq p_4^- \ll p_5^- \cdots \ll p_{n-2} - \ll p_{n-1}^- \simeq p_n^- . \end{aligned}$$

Momentum conservation (A.2) then becomes

$$\begin{aligned} 0 &= \sum_{i=3}^{n} p_{i\perp} , \\ p_2^+ &= -(p_3^+ + p_4^+) , \\ p_1^- &= -(p_{n-1}^- + p_n^-) . \end{aligned} \tag{D.4}$$

The cyclic Mandelstam invariants are

$$\begin{aligned} s &= (p_3^+ + p_4^+)(p_{n-1}^- + p_n^-), \\ s_{23} &= -(p_3^+ + p_4^+)p_3^- = -|p_{3\perp}|^2 - p_4^+ p_3^-, \\ s_{45} &= p_4^+ p_5^-, \\ &\vdots \\ s_{n-2,n-1} &= p_{n-2}^+ p_{n-1}^-, \\ s_{1n} &= -p_n^+(p_{n-1}^- + p_n^-) = -|p_{n\perp}|^2 - p_n^+ p_{n-1}^-, \end{aligned} \tag{D.5}$$

where we did not indicate s_{34} and $s_{n-1,n}$ because no approximation is taken on them. It is easy to see that

$$\begin{aligned} s_{234} &= -|p_{3\perp} + p_{4\perp}|^2, \\ s_{n-1,n,1} &= -|p_{n-1\perp} + p_{n\perp}|^2. \end{aligned} \tag{D.6}$$

D.3 Quasi-multi-Regge kinematics of a pair along the ladder

We require that the gluons are strongly ordered in rapidity, except for a pair along the ladder. In light-cone coordinates, it is

$$p_3^+ \gg p_4^+ \simeq p_5^+ \gg p_6^+; \qquad p_3^- \ll p_4^- \simeq p_5^- \ll p_6^-. \tag{D.7}$$

Momentum conservation (A.2) then becomes

$$\begin{aligned} 0 &= p_{3\perp} + p_{4\perp} + p_{5\perp} + p_{6\perp}, \\ p_2^+ &= -p_3^+, \\ p_1^- &= -p_6^-. \end{aligned} \tag{D.8}$$

The cyclic Mandelstam invariants are

$$\begin{aligned} s &= p_3^+ p_6^-, \\ s_{23} &= -p_3^+ p_3^- = -|p_{3\perp}|^2, \\ s_{34} &= p_3^+ p_4^-, \\ s_{56} &= p_5^+ p_6^-, \\ s_{61} &= -p_6^+ p_6^- = -|p_{6\perp}|^2, \end{aligned} \tag{D.9}$$

where we did not indicate s_{45} since no approximation is taken on it. The mass-shell conditions for the gluons emitted along the ladder are

$$|p_{4\perp}|^2 = \frac{s_{34}s_{46}}{s}, \qquad |p_{5\perp}|^2 = \frac{s_{35}s_{56}}{s}. \tag{D.10}$$

In addition, it is useful to evaluate

$$s_{234} = -|p_{3\perp} + p_{4\perp}|^2 - p_4^- p_5^+. \tag{D.11}$$

D.4 Quasi-multi-Regge kinematics of three-of-a-kind

In the quasi-multi-Regge kinematics of section D.3, where the outgoing gluons are emitted three in a cluster on one end and one on the other end of the ladder,

$$p_3^+ \simeq p_4^+ \simeq p_5^+ \gg p_6^+ ; \qquad p_3^- \simeq p_4^- \simeq p_5^- \ll p_6^- . \tag{D.12}$$

Momentum conservation (A.2) then becomes

$$\begin{aligned} 0 &= p_{3\perp} + p_{4\perp} + p_{5\perp} + p_{6\perp} , \\ p_2^+ &= -(p_3^+ + p_4^+ + p_5^+) , \\ p_1^- &= -p_6^- . \end{aligned} \tag{D.13}$$

The cyclic Mandelstam invariants are

$$\begin{aligned} s &= p_3^+ p_6^- , \\ s_{23} &= -(p_3^+ + p_4^+ + p_5^+) p_3^- = -|p_{3\perp}|^2 - (p_4^+ + p_5^+) p_3^- , \\ s_{56} &= p_5^+ p_6^- , \\ s_{61} &= -p_6^+ p_6^- = -|p_{6\perp}|^2 , \end{aligned} \tag{D.14}$$

where we did not indicate s_{34} and s_{45} since no approximation is taken on them. In addition, it is useful to evaluate

$$s_{234} = -|p_{3\perp} + p_{4\perp}|^2 - (p_3^- + p_4^-) p_5^+ . \tag{D.15}$$

References

[1] Z. Bern, L.J. Dixon and V.A. Smirnov, *Iteration of planar amplitudes in maximally supersymmetric Yang-Mills theory at three loops and beyond*, *Phys. Rev.* **D 72** (2005) 085001 [hep-th/0505205].

[2] Z. Bern, M. Czakon, D.A. Kosower, R. Roiban and V.A. Smirnov, *Two-loop iteration of five-point $N = 4$ super-Yang-Mills amplitudes*, *Phys. Rev. Lett.* **97** (2006) 181601 [hep-th/0604074].

[3] F. Cachazo, M. Spradlin and A. Volovich, *Iterative structure within the five-particle two-loop amplitude*, *Phys. Rev.* **D 74** (2006) 045020 [hep-th/0602228].

[4] G.P. Korchemsky and A.V. Radyushkin, *Renormalization of the Wilson loops beyond the leading order*, *Nucl. Phys.* **B 283** (1987) 342.

[5] N. Beisert, B. Eden and M. Staudacher, *Transcendentality and crossing*, *J. Stat. Mech.* (2007) P01021 [hep-th/0610251].

[6] L. Magnea and G. Sterman, *Analytic continuation of the Sudakov form-factor in QCD*, *Phys. Rev.* **D 42** (1990) 4222.

[7] G. Sterman and M.E. Tejeda-Yeomans, *Multi-loop amplitudes and resummation*, *Phys. Lett.* **B 552** (2003) 48 [hep-ph/0210130].

[8] L.F. Alday and J. Maldacena, *Comments on gluon scattering amplitudes via AdS/CFT*, *JHEP* **11** (2007) 068 [arXiv:0710.1060].

[9] L.F. Alday and J.M. Maldacena, *Gluon scattering amplitudes at strong coupling*, *JHEP* **06** (2007) 064 [arXiv:0705.0303].

[10] J.M. Drummond, J. Henn, G.P. Korchemsky and E. Sokatchev, *The hexagon Wilson loop and the BDS ansatz for the six- gluon amplitude*, *Phys. Lett.* **B 662** (2008) 456 [arXiv:0712.4138].

[11] Z. Bern et al., *The two-loop six-gluon MHV amplitude in maximally supersymmetric Yang-Mills theory*, *Phys. Rev.* **D 78** (2008) 045007 [arXiv:0803.1465].

[12] F. Cachazo, M. Spradlin and A. Volovich, *Leading singularities of the two-loop six-particle MHV amplitude*, *Phys. Rev.* **D 78** (2008) 105022 [arXiv:0805.4832].

[13] J.M. Drummond, J. Henn, G.P. Korchemsky and E. Sokatchev, *Hexagon Wilson loop = six-gluon MHV amplitude*, arXiv:0803.1466.

[14] E.A. Kuraev, L.N. Lipatov and V.S. Fadin, *Multi-Reggeon processes in the Yang-Mills theory*, *Sov. Phys. JETP* **44** (1976) 443 [*Zh. Eksp. Teor. Fiz.* **71** (1976) 840].

[15] J. Bartels, L.N. Lipatov and A.S. Vera, *BFKL Pomeron, Reggeized gluons and Bern-Dixon-Smirnov amplitudes*, arXiv:0802.2065.

[16] M.L. Mangano and S.J. Parke, *Multi-parton amplitudes in gauge theories*, *Phys. Rept.* **200** (1991) 301 [hep-th/0509223].

[17] V. Del Duca, *Parke-Taylor amplitudes in the multi-Regge kinematics*, *Phys. Rev.* **D 48** (1993) 5133 [hep-ph/9304259].

[18] V. Del Duca, *Equivalence of the Parke-Taylor and the Fadin-Kuraev-Lipatov amplitudes in the high-energy limit*, *Phys. Rev.* **D 52** (1995) 1527 [hep-ph/9503340].

[19] V. Del Duca, L.J. Dixon and F. Maltoni, *New color decompositions for gauge amplitudes at tree and loop level*, *Nucl. Phys.* **B 571** (2000) 51 [hep-ph/9910563].

[20] V. Del Duca, *Next-to-leading corrections to the BFKL equation*, in the proceedings of *Les Rencontres de Physique de la Vallee d'Aoste*, La Thuile, M. Greco ed., INFN Press, Italy (1996), hep-ph/9605404.

[21] L.N. Lipatov, *Reggeization of the vector meson and the vacuum singularity in nonabelian gauge theories*, *Sov. J. Nucl. Phys.* **23** (1976) 338 [*Yad. Fiz.* **23** (1976) 642].

[22] L.N. Lipatov, *High-energy scattering in QCD and in quantum gravity and two-dimensional field theories*, *Nucl. Phys.* **B 365** (1991) 614.

[23] V.S. Fadin and L.N. Lipatov, *Radiative corrections to QCD scattering amplitudes in a multi-Regge kinematics*, *Nucl. Phys.* **B 406** (1993) 259.

[24] V. Del Duca and C.R. Schmidt, *Virtual next-to-leading corrections to the impact factors in the high-energy limit*, *Phys. Rev.* **D 57** (1998) 4069 [hep-ph/9711309].

[25] V. Del Duca and C.R. Schmidt, *Virtual next-to-leading corrections to the Lipatov vertex*, *Phys. Rev.* **D 59** (1999) 074004 [hep-ph/9810215].

[26] V. Del Duca and E.W.N. Glover, *Testing high-energy factorization beyond the next-to-leading-logarithmic accuracy*, *JHEP* **05** (2008) 056 [arXiv:0802.4445].

[27] Z. Bern, V. Del Duca and C.R. Schmidt, *The infrared behavior of one-loop gluon amplitudes at next-to-next-to-leading order*, *Phys. Lett.* **B 445** (1998) 168 [hep-ph/9810409].

JHEP12(2008)097

[28] V.S. Fadin and R. Fiore, *Quark contribution to the gluon-gluon-Reggeon vertex in QCD*, *Phys. Lett.* **B 294** (1992) 286.

[29] V.S. Fadin, R. Fiore and A. Quartarolo, *Radiative corrections to quark quark reggeon vertex in QCD*, *Phys. Rev.* **D 50** (1994) 2265 [hep-ph/9310252].

[30] V.S. Fadin, M.I. Kotsky and R. Fiore, *Gluon Reggeization in QCD in the next-to-leading order*, *Phys. Lett.* **B 359** (1995) 181.

[31] V.S. Fadin, R. Fiore and A. Quartarolo, *Reggeization of quark quark scattering amplitude in QCD*, *Phys. Rev.* **D 53** (1996) 2729 [hep-ph/9506432].

[32] V.S. Fadin, R. Fiore and M.I. Kotsky, *Gluon Regge trajectory in the two-loop approximation*, *Phys. Lett.* **B 387** (1996) 593 [hep-ph/9605357].

[33] J. Blumlein, V. Ravindran and W.L. van Neerven, *On the gluon Regge trajectory in $\mathcal{O}(\alpha_s^2)$*, *Phys. Rev.* **D 58** (1998) 091502 [hep-ph/9806357].

[34] V. Del Duca and E.W.N. Glover, *The high energy limit of QCD at two loops*, *JHEP* **10** (2001) 035 [hep-ph/0109028].

[35] A.V. Kotikov and L.N. Lipatov, *NLO corrections to the BFKL equation in QCD and in supersymmetric gauge theories*, *Nucl. Phys.* **B 582** (2000) 19 [hep-ph/0004008].

[36] A.V. Kotikov and L.N. Lipatov, *DGLAP and BFKL evolution equations in the $N = 4$ supersymmetric gauge theory*, *Nucl. Phys.* **B 661** (2003) 19 [*Erratum ibid.* **685** (2004) 405] [hep-ph/0208220].

[37] J.M. Drummond, G.P. Korchemsky and E. Sokatchev, *Conformal properties of four-gluon planar amplitudes and Wilson loops*, *Nucl. Phys.* **B 795** (2008) 385 [arXiv:0707.0243].

[38] S.G. Naculich and H.J. Schnitzer, *Regge behavior of gluon scattering amplitudes in $N = 4$ SYM theory*, *Nucl. Phys.* **B 794** (2008) 189 [arXiv:0708.3069].

[39] Z. Bern, M. Czakon, L.J. Dixon, D.A. Kosower and V.A. Smirnov, *The four-loop planar amplitude and cusp anomalous dimension in maximally supersymmetric Yang-Mills theory*, *Phys. Rev.* **D 75** (2007) 085010 [hep-th/0610248].

[40] F. Cachazo, M. Spradlin and A. Volovich, *Four-loop cusp anomalous dimension from obstructions*, *Phys. Rev.* **D 75** (2007) 105011 [hep-th/0612309].

[41] M. Spradlin, A. Volovich and C. Wen, *Three-loop leading singularities and BDS ansatz for five particles*, *Phys. Rev.* **D 78** (2008) 085025 [arXiv:0808.1054].

[42] C. Anastasiou, Z. Bern, L.J. Dixon and D.A. Kosower, *Planar amplitudes in maximally supersymmetric Yang-Mills theory*, *Phys. Rev. Lett.* **91** (2003) 251602 [hep-th/0309040].

[43] V. Del Duca, C. Duhr, E.W.N. Glover and V.A. Smirnov, *The two-loop five-point amplitude in the high-energy limit*, in progress.

[44] J.M. Drummond, J. Henn, G.P. Korchemsky and E. Sokatchev, *Conformal Ward identities for Wilson loops and a test of the duality with gluon amplitudes*, arXiv:0712.1223.

[45] R.C. Brower, H. Nastase, H.J. Schnitzer and C.-I. Tan, *Implications of multi-Regge limits for the Bern-Dixon-Smirnov conjecture*, arXiv:0801.3891.

[46] R.C. Brower, H. Nastase, H.J. Schnitzer and C.-I. Tan, *Analyticity for Multi-Regge limits of the Bern-Dixon-Smirnov amplitudes*, arXiv:0809.1632.

[47] V. Del Duca, *Real next-to-leading corrections to the multigluon amplitudes in the helicity formalism*, *Phys. Rev.* **D 54** (1996) 989 [hep-ph/9601211].

[48] V.S. Fadin and L.N. Lipatov, *Next-to-leading corrections to the BFKL equation from the gluon and quark production*, *Nucl. Phys.* **B 477** (1996) 767 [hep-ph/9602287].

[49] V. Del Duca, A. Frizzo and F. Maltoni, *Factorization of tree QCD amplitudes in the high-energy limit and in the collinear limit*, *Nucl. Phys.* **B 568** (2000) 211 [hep-ph/9909464].

[50] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *One-loop n-point gauge theory amplitudes, unitarity and collinear limits*, *Nucl. Phys.* **B 425** (1994) 217 [hep-ph/9403226].

[51] A. Brandhuber, B. Spence and G. Travaglini, *From trees to loops and back*, *JHEP* **01** (2006) 142 [hep-th/0510253].